Cambridge Assessment International Education

Teaching Pack

3.7 Vectors

Cambridge International AS & A Level Mathematics 9709

Version 1

Cambridge Pathway

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Introduction

This pack will help you to develop your learners' skills in mathematical thinking and mathematical communication, which are essential for success at AS & A Level and in further education.

Mathematical thinking and communication will be developed by focusing on:

1. Conceptual understanding – the 'why' behind the 'what'
2. Strategic competence – forming and solving problems
3. Adaptive reasoning – explanations, justifications and deductive reasoning

Throughout all activities, the learners will also develop:

- procedural fluency – know when, how and which rules to use
- positive disposition – believe maths can be learned, applied and is useful
- their skills in writing mathematically – writing working & proofs

These link to the course Assessment Objectives (AOs) which you can find in detail in the syllabus:

A01 Knowledge and understanding

A02 Application and communication

Each *Teaching Pack* contains one or more lesson plans and associated resources, complete with a section of preparation and reflection.

Each lesson is designed to be an hour long but you should adjust the timings to suit the lesson length available to you and the needs of your learners.

Important note
Our *Teaching Packs* have been written by **classroom teachers** to help you deliver topics (but not necessarily a whole topic) and skills that can be challenging. Use these materials to supplement your teaching and engage your learners. You can also use them to help you create lesson plans for other topics.

This content is designed to give you and your learners the chance to explore a more active way of engaging with mathematics that encourages independent thinking and a deeper conceptual understanding. It is not intended as specific practice for the examination papers.

The *Teaching Packs* are designed to provide you with some example lessons of how you might deliver content. You should adapt them as appropriate for your learners and your centre. A single pack will only contain at most four lessons, it will **not** cover a whole topic. You should use the lesson plans and advice provided in this pack to help you plan the remaining lessons of the topic yourself.

Lesson preparation

This *Teaching Pack* will cover the following syllabus content:

Candidate should be able to:	Notes and examples
• use standard notation for vectors, i.e. $\begin{pmatrix} x \\ y \end{pmatrix}$, $x\mathbf{i} + y\mathbf{j}$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\overrightarrow{AB}$, $\mathbf{a}$	
• carry out addition and subtraction of vectors and multiplication of a vector by a scalar, and interpret these operations in geometric terms	e.g. 'OABC is a parallelogram' is equivalent to $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$. The general form of the ratio theorem is not included, but understanding that the midpoint of AB has position vector $\frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$ is expected.
• understand the significance of all the symbols used when the equation of a straight line is expressed in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, and find the equation of a line, given sufficient information	e.g. finding the equation of a line given the position vector of a point on the line and a direction vector, or position vectors of two points on the line.
• determine whether two lines are parallel, intersect or are skew, and find the points of intersection of two lines when it exists	Calculation of the shortest distance between two skew lines is not required. Finding the equation of the common perpendicular to two skew lines is also not required.

The remaining *two* bullet points for topic 3.7 Vectors are not covered in this *Teaching Pack* (see the syllabus for detail). You will need to write your own lesson plans for these items.

Candidate should be able to:	Notes and examples
• calculate the magnitude of a vector, and use unit vectors, displacement vectors and position vectors	In 2 or 3 dimensions.
• use formulae to calculate the scalar product of two vectors, and use scalar products in problems involving lines and points	e.g. finding the angle between two lines, and finding the foot of the perpendicular from a point to a line; questions may involve 3D objects such as cuboids, tetrahedra (pyramids), etc. Knowledge of the vector product is not required.

Dependencies

For all lesson plans in this *Teaching Pack*, knowledge of the content of section 1 Pure Mathematics 1 of the 9709 syllabus is assumed and learners may be required to use such knowledge in answering questions.

Prior knowledge and skills

For all lessons, it is assumed that learners have already completed Cambridge IGCSE™ Mathematics 0580, or a course at an equivalent level. See the syllabus for more details of the expected prior knowledge for taking Cambridge International AS & A Level Mathematics 9709.

When planning any lesson, make a habit of always asking yourself the following questions about your learners' prior knowledge and skills:

- Do I need to re-teach this or do learners just need some practice?
- Is there an interesting activity that will efficiently achieve this?

Key learning objectives

The following list represents the main underlying concepts that you should make sure your learners have understood by the end of this topic:

- There are several ways to describe and notate a vector depending on what the vector is being used for and where it starts (i.e. position vectors start at the origin).
- Addition and subtraction of vectors happens component-wise.
- One form of multiplication of vectors is the scalar product.
- **a** . **b** = |**a**| |**b**| cos θ, where θ is the angle made between the direction vectors **a** and **b**.
- We can represent lines In 2D and 3D using the vector equation **r** = **a** + t**b**.
- Two lines are parallel when their direction vectors are parallel.
- Two lines are perpendicular when their direction vectors are perpendicular.

Why this topic matters

The topic of vectors introduces new ways to work with coordinate geometry questions and builds a foundational understanding of working with vectors that is useful for 4.1 Forces and equilibrium, Mechanics (Paper 4) component of the course.

Key terminology and notation

Your learners will need to be confident with the following terminology and notation.

displacement vector	a vector which represents the translations between two points e.g. $\overrightarrow{AB}$
magnitude of a vector	the length of a vector e.g. **a** = x**i** + y**j** + z**k**, then |**a**| = $\sqrt{x^2 + y^2 + z^2}$.
position vector	a vector which represents the position of a point with respect to the origin e.g. $\overrightarrow{OA}$
r = **a** + *t***b**	a vector equation for a straight line, where **r** is the position vector of points on the line, **a** is a position vector of a point on the line and **b** is a direction vector in the direction of the line
scalar product a.b	also known as the dot product. Let **a** = x_1**i** + y_1**j** + z_1**k** and **b** = x_2**i** + y_2**j** + z_2**k** then **a . b** $\equiv x_1x_2 + y_1y_2 + z_1z_2$
unit vector	a vector of length 1

Insights video

There is an Insights video linked to this *Teaching Pack*:

- **3.7 Vectors** – watch this video which will show some of the misunderstandings learners have when solving vector equations.

Teacher tutorials

There are *two* tutorials linked to this *Teaching Pack*:

- **Vector notation**
  Review this tutorial before teaching Lesson plan 1; this will show you how to highlight the different notations and terminology surrounding vectors.
- **Vector equations of lines**
  Review this tutorial before teaching Lesson plan 2; this will show you how to connect the ideas of straight lines from IGCSE (or equivalent) and 1.3 Coordinate geometry with vectors.

Lesson progression

Lesson 1 covers the first two bullet points of syllabus content. Lesson 2 focuses on developing and working with vector equations of straight lines. The ideas of Lessons 1 and 2 are then used in Lesson 3.

Going forward

This topic links to 4.1 Forces and equilibrium Mechanics (Paper 4) component of the course.

Lesson plan 1: Vector notation and basics

Preparation	• Review the Teacher tutorial *Vector notation.*

Resources	• Worksheet A: *Match the vectors* • Worksheet B: *Vector paths* • Lesson slides *Vector notation* • Paper, Mini whiteboards or other writing materials

Learning objectives	By the end of the lesson: • ***all*** learners should be able to read and interpret different vector notations. • **most** learners should be able to answer geometry problems using vector notations • **some** learners should be able to prove confidently geometric facts using vectors.

Dependencies

Learners need to know how to work with 2D and 3D coordinates. Learners should also have a knowledge of graph transformations in the form of translations from IGCSE (or equivalent) and of straight line graphs from 1.3 Coordinate geometry.

Common misconceptions

Misconception	Problems this can cause	An example way to resolve the misconception
The difference between a position vector and a displacement vector.	If learners do not know the difference then they will not be able to interpret worded questions nor will they be able to understand the construction of a vector equation of a line.	Modelling the correct language with learners will help reinforce the correct definition for each type of vector.
Not understanding that vector paths are equivalent if they have the same start point and end point as each other.	If learners do not see that all vector paths that have the same start point and end point are equivalent, they will not be able to solve problems involving diagrams.	Using a regular hexagon, label all the vertices A to F and the centre O. Ask learners to state possible paths that go from A to F. Then label two non-parallel edges with vectors **a** and **b**. Ask learners to describe each of their paths now in terms of their paths. Highlight that, if we simplify each of the vector paths, they all give the same simplified vector path. (This resource is used in the second half of the main lesson.)

Timings	Activity
5 min	**Starter/Introduction** Lesson slides *Vector notation* Give out a piece of A4 plain paper per pair and ask learners to write the following in the centre:

Timings	Activity

A vector is…

5 min

Ask learners to work in their pairs and write down anything that they can remember from IGCSE (or equivalent) that connects to this start of a statement.

Learners may remember the following: vectors are used to describe movement between two points; vectors are used to describe a transformation of a shape or curve; vectors have a magnitude; vectors have a direction.

After 5 minutes, ask two pairs to join up and review each other's notes. While learners are doing this, go around each group of four and spot any common themes.

Finish this introduction by collecting together the learners' ideas together on the main board (slide 2) and summarise their work. Conclude by finishing the statement: A vector is a quantity that has a magnitude and a direction.

Main lesson

5 min

This part of the lesson is about vector notation and it may be useful to start with an example of each of the following before learners use Worksheet A. The introductory activity will allow you to determine if this will be required and the Lesson slides *Vector notation* slide 3 and 4 could be used with learners here.

Support: (slide 5) A vector is a quantity that has a magnitude and a direction. We can represent a vector in several ways:

- as a movement in the *x* and *y* directions e.g. a column vector
- as an arrow (a directed line segment)
- as a movement between two identified points e.g. $\overrightarrow{AB}$ is the vector that describes the movement from A to B.

10 min

Worksheet A: *Match the vectors*

Give out Worksheet A to each learner (you can also use Lesson slides *Vector notation* slides 6 to 8. You could use the introduction activity to pair learners up so that those who did not remember as much about their previous work on vectors are with a learner who remembers more.

Firstly, ask learners to match up the column vectors with the arrows in the grid.

Learners should find that they can get started on this task quickly. Most learners will write each arrow as a column vector and match them that way. However, some learners may notice that the arrows can be classified by the direction in which they point and use this to help sort the column vectors.

Secondly, ask learners to match up the column vector/arrows they have with the movement between the points on the coordinate axes grid.

Most learners will write each position vector of the points given and match them that way. That will leave a few vectors that do not match and so they will then start to look at the movement between the points, the direction vectors.

Timings	Activity

Finally, ask learners to read the final part of the sheet 'Movement between points'. Inform learners that they are calculating the **resultant vector** for each movement.

5 min

Ask learners to gather into another group of four and review their answers with each other (Display Worksheet A: Answers if required – slides 7 and 8)

You can use the following questions to frame a discussion with learners:

Question	Ideas
A position vector of a point is a vector that describes the movement from the origin to the point. Which vectors in Worksheet A are position vectors?	This will introduce the terminology of position vector.
We can describe all vectors as direction vectors, since vectors determine a direction to move in. Which vectors in Worksheet A are direction vectors that are not position vectors?	Using this question, you can start to help learners make the distinction between position vectors and all direction vectors.

5 min

Introduce learners to the idea of **i**, **j** and **k** (you should highlight the 3D aspect of using k) as another way to describe a vector based on these special vectors. Highlight that the length of each vector is 1, so we call them unit vectors.

i = $\begin{pmatrix}1\\0\\0\end{pmatrix}$, **j** = $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and **k** = $\begin{pmatrix}0\\0\\1\end{pmatrix}$ is how we write them as column vectors.

Ask learners to write $\overrightarrow{OA}$, $\overrightarrow{EC}$ from the movement between points section, using the unit vectors **i**, **j** and **k**.

10 min

Lesson slides *Vector notation* slide 10
Worksheet B(i): *Vector paths*
Give out Worksheet B Part I and ask the following questions for learners to work on together in pairs.

Question	Ideas
Label $\overrightarrow{AB}$ **a** and $\overrightarrow{OA}$ **b**. What is the vector: $\overrightarrow{AE}$ $\overrightarrow{DB}$ in terms of the vectors a and b? What do you notice about these vectors?	The vectors here are parallel. Ask learners what is true about vectors when they are parallel. Can they identify other vectors that are parallel in this diagram?
Describe as many ways to form the vector $\overrightarrow{OA}$ as possible. What do they all have in common?	All resultant vectors of the paths will be equal.

Learners may need some assistance in answering the questions. If another learner has identified something useful for each point, ask them to share it with the class.

Timings	Activity
5 min	Collect learners' ideas in a class discussion and highlight the following points: 1) Paths with the same start point and end point as each other are equivalent and so the **resultant vectors** will be equal. 2) Vectors that are multiples of each other are parallel.
10 min	**Plenary** Lesson slides *Vector notation* slide 11 Worksheet B(ii): *Vector paths* Using Worksheet B Part II, ask learners to apply their knowledge of vector notation and the ideas collected in this lesson to answer the questions. Learners are expected to share their solutions with the class. **Support:** To help learners, the following prompts can be used for each question: 1) How can we describe the paths in this question using the points in our diagram? Can we label any paths that are the same as the vectors **a** and **b**? 2) How do we find the vector $\overrightarrow{AO}$? How do we find the vector $\overrightarrow{OE}$? 3) What does it mean to be half-way between C and E in our diagram? Can you mark on the point P? 4) What does it mean for N to split the edge ED in the ration 2:3? How would we split the edge if the total length was 5, 10 and 1 units long?

Reflection Reflect on your lesson; use the **Lesson reflection** notes to help you.

Lesson plan 2: Vector equations of lines

Preparation Review the Teacher tutorial *Vector equations of lines.*

Resources

- Worksheet C: *Making lines*
- Worksheet D: *Analysis of vector equations of lines*
- Worksheet E: *Vector equation of a line*
- Paper, Mini whiteboards or other writing materials
- Lesson slides: *Vector equations of lines*
- Video: *Vectors and lines*
- GeoGebra and VectorsandLines.ggb

Learning objectives

By the end of the lesson:

- ***all*** learners should be able to identify the and use the components of a vector equation of straight line
- **most** learners should be able to construct a vector equation of a straight line
- **some** learners should be able to connect the various forms for an equation of a line

Dependencies

Learners need to know how to find the gradient from two pairs of coordinates. They should also be familiar with equations in the form $y = mx + c$, $ax + by = c$ and $y – a = m(x - b)$.

Common misconceptions

Misconception	Problems this can cause	An example way to resolve the misconception
Learners confuse the vector equation of a straight line in 2D and 3D with the cartesian equation of a line in 2D.	This can lead to learners not able to identify what the 'gradient' of the straight line is as they are looking for a single value to describe this. Learners are also not aware that there an infinite number of vector equations for a single line.	This will be covered in the lesson content below. The introductory task will identify the key features of the Cartesian form of a straight line graph and the key idea of the movement that the gradient represents.
Learners are unsure how to deal with a parameter.	If learners do not know what the parameter in a vector equation of a straight line is doing, they will struggle to solve intersection problems with this object.	This will be covered in the lesson content below. Worksheets C, D and E will build on the ideas given in the video and will help to address this potential barrier of what a parameter is and how to us it.

Timings	Activity
10 min	**Starter/Introduction** Lesson slides: *Vector equations of lines* slides 2-3 This activity is a refresher on the ideas of straight line graphs in 2D from IGCSE (or equivalent). Display the equations: $y = 3x - 4$

Timings	Activity
	$6x – 3y = 4$ $y + 1 = 3(x-1)$ on a projector or written on a board. Ask learners the following questions: 1) What do the equations represent? 2) Are the equations connected? 3) What are the key features of each equation and how does that help define what the equations represent? Allow learners a moment to think about the questions individually before they speak in pairs. Then ask two pairs to join and discuss their thoughts. Ask each group to come up with a couple of key points for each equation. Listen to learners' responses and highlight the key point of: Gradient of each line is determined by a movement e.g. as you move 1 in the positive x-direction, you move m in the positive y-direction to stay on the line for y=mx+c. Make the connection that movement in the coordinate directions can be described as a vector.
	Main lesson
10 min	Show learners the video Vectors and lines. Play the video for a second time and ask learners to write down on mini whiteboards what the key ideas are. Learners should notice that we are identifying a direction vector for the line which defines the gradient of the line. Learners may need help in translating the movement they see as a single value (the 'm' in y=mx+c) as a vector quantity when we consider what is actually happening i.e. as you move 1 in the positive x-direction you move m in the positive y-direction. Learners may find it initially challenging to see that the 'y-intercept' can be seen as the position vector of a point on the line. Help learners formulate written notes based on the video, identifying the key ideas highlighted in the teacher tutorial. Ask learners to check that their notes reflect the key elements of a vector equation. You could do this by asking learners to read each other's notes to check that the key points are identified.
15 min	Lesson slides: *Vector equations of lines* slide 4 Worksheet C: *Making lines* Give out Worksheet C and ask learners in pairs to construct their own vector equation for the questions given, writing their ideas down on Worksheet C. Some learners may find it challenging to focus on what is required for each equation. Allow learners to work in pairs to help formulate their ideas and emphasis any good conversations you hear in each pair by asking learners to share their ideas with other pairs that require some additional support. **Support:** Ask learners to identify what a position vector and what a direction vector are for each line. Model this with some questions that you generate yourself using Worksheet C.
15 min	Lesson slides: *Vector equations of lines* slide 5 Worksheet D: *Analysis of vector equations of lines*

Timings	Activity
	We will now work with lines in 3D. Use the first equation in Worksheet D with the GeoGebra file (VectorsandLines.ggb) to describe the features of vector equations representing lines in 3D. Give out Worksheet D and ask learners to determine which equations represent the same lines, which equations represent parallel lines and which equations are neither parallel nor the same line. Learners will try and apply their ideas from 2D to 3D. This will enable you as the teacher to identify those who are correctly using the direction and position vectors to describe each line. Ideas to highlight during this activity: Parallel lines will have direction vectors that are parallel. You can have more than one vector equation to describe a line. Non-parallel lines can intersect in 3D but they can also not intersect. These lines are called skew lines.
	**Plenary** *Worksheet E: Vector equation of a line* Give out Worksheet E. Ask learners to identify what will happen if they change the different components of the vector equation given. Use GeoGebra with the command Line[(2,-1),Vectors[(1,5)]] and introduce sliders to see what happens when you change the components.

Reflection Reflect on your lesson; use the **Lesson reflection** notes to help you.

Lesson plan 3: Problem solving with vectors and lines

Resources	• Paper, Mini whiteboards or other writing materials • Worksheet F: *Making quadrilaterals* • GeoGebra and MakingQuad.ggb

Learning objectives	By the end of the lesson: • ***all*** learners should be able to solve problems involving intersections of vector equations of straight lines • **most** learners should be able to solve problems involving the construction of vector equations of straight lines and geometric properties e.g. parallel and perpendicular lines • **some** learners should be able to understand proofs involving vector equations of straight lines

Dependencies

Learners need to know how to find the gradient from two pairs of coordinates. They should also be familiar with equations in the form $y = mx + c$.

Common misconceptions

Learners may still demonstrate the misconceptions and errors from lessons 1 and 2. This lesson is a further chance to diagnose these misconceptions and address them. Learners do find unstructured questions challenging in all subjects and so learners will need to think through the Problem Solving framework when consolidating their knowledge in this lesson.

Timings	Activity
10 min	**Starter/Introduction** Lesson slides: *Problem solving with vectors and lines* slide 2 Display the Vector squares slide and ask learners to focus on the question: *Can you convince yourself that the adjacent edges are perpendicular?* Collect learners' ideas on a board that the whole class can see. Learners will need to look back at their ideas from coordinate geometry about perpendicular lines and apply that idea here in the context of vector equations. Some learners may see this in a geometric way and produce diagrams based on the direction vectors of the non-parallel sides. **Note:** It is not a requirement that learners have met the scalar product before this lesson as the work is in 2D. This lesson could be used as a way to highlight the need for a better process to determine whether two lines are perpendicular, as the methods described here will only work in 2D.
5 min	**Main lesson** Ask learners what the key properties of a square are and how we can determine those key properties using our work on vectors.

Timings	Activity
	Learners may have forgotten that squares have equal edge lengths and their angles are right angles. It may take learners a few moments to connect these ideas from geometry with the use of vectors e.g. distance between two points is the length of a vector.
5 min	**Support:** Allow learners to use the GeoGebra MakingQuad.ggb to help them 'see' what is happening. Ask learners what would need to happen to the lines so that the enclosed space is a rectangle. How would this change affect the vector equations we have been working with? Ask learners to identify what specifically would need to change in their original four equations in order for the enclosed space to be a rectangle. Is there only one way to do this? Learners may initially struggle to know what to change in the vector equations but they may have an idea using the Cartesian form. It is important that learners are connecting any ideas they form from the Cartesian equation of each line with the vector equations that define them in this situation.
10 min	**Support:** Allow learners to use the GeoGebra MakingQuad.ggb to help them 'see' what is happening. Ask learners what would need to happen to the lines if we wanted to form a: 1) rhombus 2) parallelogram Worksheet F: *Making quadrilaterals* Give out Worksheet F and ask learners to try and adapt the original vector equations to form these new shapes. Ask learners to highlight on the sheet what is changed each time in the components of the vector equation of the line and how they know what the effect will be geometrically. For each quadrilateral, they should explain: • how you changed your vector equations • how you knew what the effect of the change was going to produce • how you can verify that the shape you have created is the quadrilateral you require. **Support:** Learners may need a reminder as to what the definition of each of the quadrilaterals is, and how their properties can be determined using knowledge of the vectors that define their vertices.
15 min	**Challenge:** Ask learners to complete the task in 3 dimensions. Ask learners to share their ideas with a partner. Then challenge them to create their own vector equations that define an enclosed quadrilateral. Each pair can then give their vector equations to another pair and ask them to prove which quadrilateral their enclosed area gives. Answers can be checked quickly by using GeoGebra or by having each pair come to the front of the class and present their proof of the shape they have.
15 min	**Plenary** Ask learners to write down the key ideas that have been used in the lesson and how they were applied in each situation.

Timings	Activity
	If learners are struggling to write down the key ideas you could write down the following list and ask them to provide an example from the lesson of where they have used that idea.

Idea	Example
Vector equations of parallel lines have direction vectors that are parallel.	*The lines that made up the pairs of parallel sides needed this property to be checked.*
To find intersections of lines we solve linear equations for the parameter for each x, y and z component (in 3D).	*The intersection of the non-parallel lines gave the vertices of the enclosed shape.*
The length of a vector is called its magnitude and we can use Pythagoras' Theorem to calculate it.	*The length of the vector between the vertices of the enclosed shape was used to check we have the correct quadrilateral.*
Lines are perpendicular in 2D if the gradients are negative reciprocals. How do we determine this in 3D? **Optional:** Lines are perpendicular if the direction vectors of the lines have a dot product of zero.	*The angles in the enclosed square and rectangle needed to be 90 degrees.*

Reflection Reflect on your lesson; use the **Lesson reflection** notes to help you.

Planning your own lessons

You now need to plan lessons to cover the following bullet points:

Candidate should be able to:	Notes and examples
• calculate the magnitude of a vector, and use unit vectors, displacement vectors and position vectors	In 2 or 3 dimensions.
• use formulae to calculate the scalar product of two vectors, and use scalar products in problems involving lines and points	e.g. finding the angle between two lines, and finding the foot of the perpendicular from a point to a line; questions may involve 3D objects such as cuboids, tetrahedra (pyramids), etc. Knowledge of the vector product is not required.

Follow the structure of the *Teaching Pack*, and use techniques from the 'How to' guides, to create your own engaging lessons to cover these bullet points. Consider what preparation you need for each lesson: what prior knowledge is needed, what are the key objectives, what are the dependencies, what common misconceptions are there, and so on.

Below, we have provided an outline of some activities and approaches you might like to try.

Lesson 4: Applying knowledge of vectors
Common misconceptions: Learners can be unsure where to start with a problem and so you will need to build up some strategies to help them by exposing them to unfamiliar situations.
Starter: You could try https://nrich.maths.org/7460 as a way to practise fluency
Main: You could use the following more challenging vector questions from Cambridge Assessment (hosted on Underground Mathematics) and have a class discussion about the ideas required for each question.
https://undergroundmathematics.org/vector-geometry/r8704
https://undergroundmathematics.org/vector-geometry/r6007
https://undergroundmathematics.org/vector-geometry/r5992
https://undergroundmathematics.org/vector-geometry/r5153
https://undergroundmathematics.org/vector-geometry/r8339
https://undergroundmathematics.org/vector-geometry/r8215
Plenary: You could try asking learners to create a revision card, either by identifying the types of mathematical problems where vectors are used or by writing their own question to test the skills they have encountered in the previous lessons.

Lesson 5: The scalar product
Common misconceptions: Learners can confuse the definition of the scalar product ($\mathbf{a} \,.\, \mathbf{b} := \sum a_i b_i$) with the connection with the cosine of the angle between the two vectors ($\mathbf{a} \,.\, \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$). Using the correct language here will help learners understand the difference between a definition and the equality.
Starter: You could try https://nrich.maths.org/7787&part=note as a reminder of the cosine rule.
Main: You could use resource https://nrich.maths.org/2393 and have a class discussion about what the scalar product is and what geometric relationship it corresponds to. You could then use https://nrich.maths.org/6576 to gain a greater understanding of what is happening in 3D. For learners who are interested you could use https://nrich.maths.org/6451 as an application of this idea.
Plenary: You could try asking learners to create a revision card, identifying the types of mathematical problems where the scalar product is used and the type of mathematical problems where the connection between the scalar product and the cosine of the angle between two vectors is used.

You will find some other activity suggestions in the Scheme of Work.

Lesson reflection

As soon as possible after the lesson you need to think about how well it went.

One of the key questions you should always ask yourself is:

Did all learners get to the point where they can access the next lesson? If not, what will I do?

Reflection is important so that you can plan your next lesson appropriately. If any misconceptions arose or any underlying concepts were missed, you might want to use this information to inform any adjustments you should make to the next lesson.

It is also helpful to reflect on your lesson for the next time you teach the same topic. If the timing was wrong or the activities did not fully occupy the learners this time, you might want to change some parts of the lesson next time. There is no need to re-plan a successful lesson every year, but it is always good to learn from experience and to incorporate improvements next time.

To help you reflect on your lesson, answer the most relevant questions below.

Were the lesson objectives realistic?
What did the learners learn today? Or did they learn what was intended? Why not?
What proportion of the time did we spend on the most important topics?
Were there any common misconceptions?
What do I need to address next lesson?
What was the learning atmosphere like?
Did my planned differentiation work well?
How could I have helped the lowest achieving learners to do more?
How could I have stretched the highest achieving learners even more?
Did I stick to timings?
What changes did I make from my plan and why?

Summary evaluation

What two things went really well? (Consider both teaching and learning.)

What two things would have improved the lesson? (Consider both teaching and learning.)

What have I learned from this lesson about the class or individuals that will inform my next lesson?

Worksheets and answers

	Worksheet	Answers
For use with *Lesson 1*:		
A: Match the vectors	**21**	**28**
B(i): Vector paths	**22**	
B(ii): Vector paths	**23**	**29**
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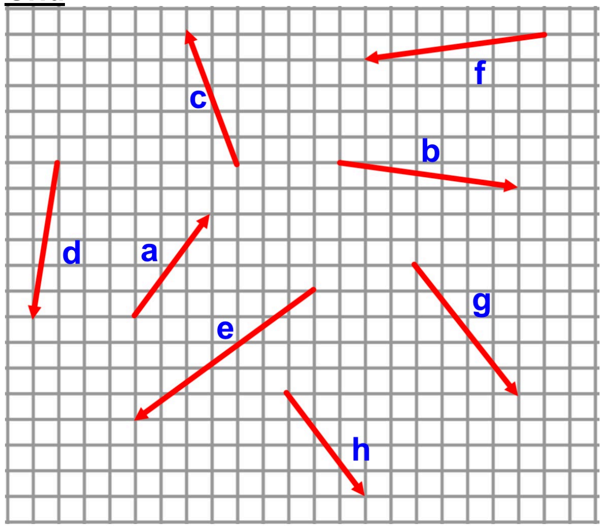
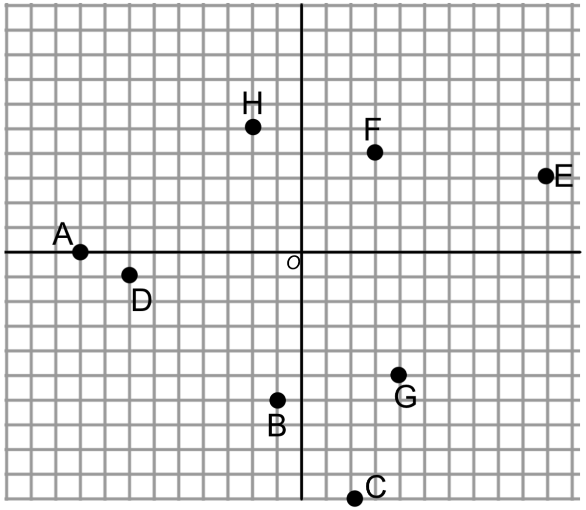
Worksheet A: Match the vectors

Match up the column vector with an arrow from the grid. Then determine which vectors describe the movement between the points A to H in the coordinate axis grid.

Grid



Coordinate axes



Column vectors

$$\begin{pmatrix}-1\\-6\end{pmatrix} \quad \begin{pmatrix}3\\4\end{pmatrix} \quad \begin{pmatrix}7\\-1\end{pmatrix} \quad \begin{pmatrix}-2\\5\end{pmatrix} \quad \begin{pmatrix}-7\\-5\end{pmatrix} \quad \begin{pmatrix}3\\-4\end{pmatrix} \quad \begin{pmatrix}4\\-5\end{pmatrix} \quad \begin{pmatrix}-7\\-1\end{pmatrix}$$

Movement between points

Once you have matched all the given vectors to the points in the coordinate axis grid, find the following vector paths (i) as a simplified column vector, (ii) using the vectors named in the Grid section.

1) $\overrightarrow{OA}$

2) $\overrightarrow{OE}$

3) $\overrightarrow{OC}$

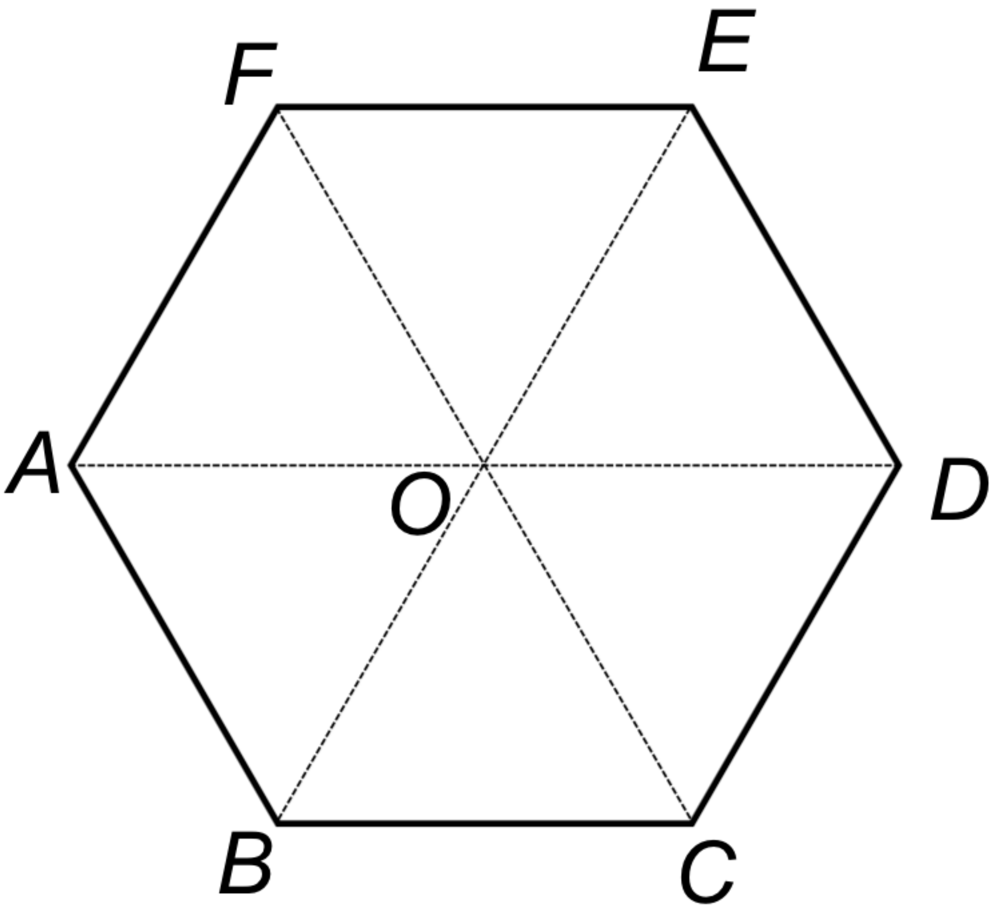
4) $\overrightarrow{AE}$

5) $\overrightarrow{CA}$

6) $\overrightarrow{EC}$

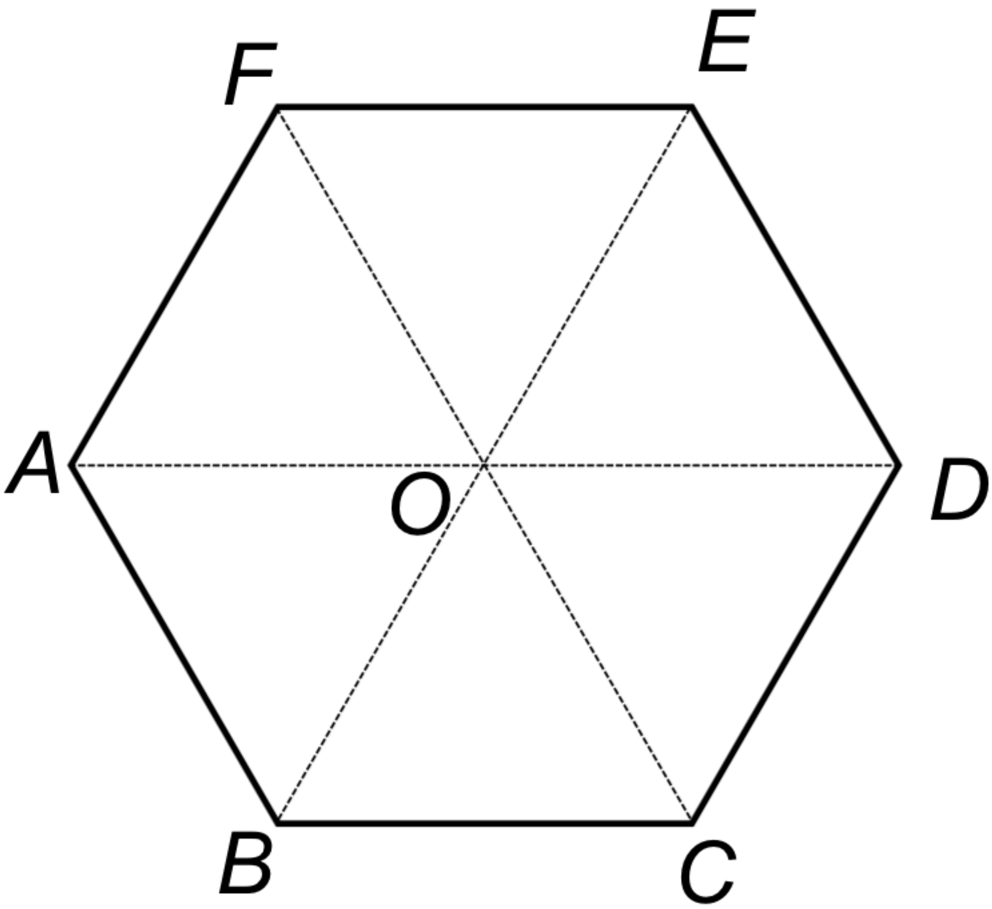
Worksheet B(i): Vector paths

Part I



Worksheet B(ii): Vector paths

Part I



Part II

Using the diagram above, answer the following questions. Let $\overrightarrow{AF}$ = **a** and $\overrightarrow{BC}$ = **b**.

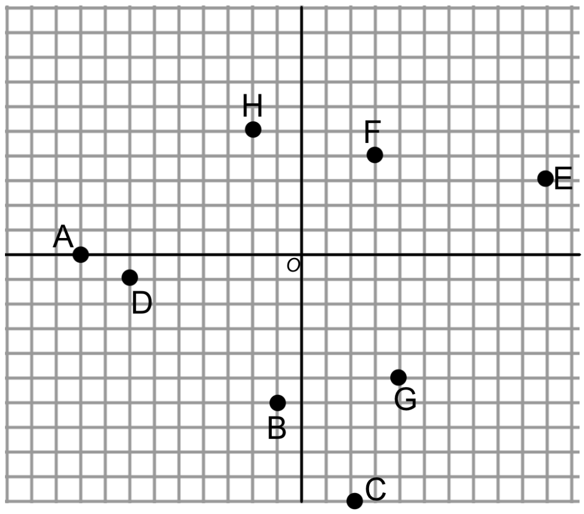
1) Show that $\overrightarrow{AE}$ is parallel to $\overrightarrow{DB}$.

2) The point M is half-way between O and E. Find the vector $\overrightarrow{AM}$ in terms of the vectors **a** and **b**.

3) The point P is half-way between C and E. Find the vector $\overrightarrow{MP}$ in terms of the vectors **a** and **b**.

4) The point N is along ED so that N splits ED in the ratio 2:3 (i.e. EN:ND = 2:3). Find the vector $\overrightarrow{BN}$ in terms of the vectors **a** and **b**.

Worksheet C: Making lines



Can you construct a vector equation for the following lines?
Write down how you got your equation and any choices that you made along the way.

1) A line that passes through the origin O and the point F.

2) A line that passes through the points B and D.

3) A line that passes through E and is in the direction of $\overrightarrow{BG}$.

4) A line that passes through C and is parallel to the *y*-axis.

5) A line that passes through C and is parallel to the *x*-axis.

6) A line that passes through D and is parallel to $y=-x$.

7) A line that passes through H and is perpendicular to $y=2x+3$.

8) A line that passes through A and is parallel to $4x+5y=10$.

Worksheet D: Analysis of vector equations of lines

The vector equations below describe lines in 3D. Can you determine if:
1) any of the equations define lines that are parallel to each other?
2) any of the equations define the same line?
3) any of the equations define lines that intersect?
4) any of the equations define lines that are parallel to any of the coordinate axes?
While answering these questions, write down how you came to these conclusions and what features of the equations that you are working with that led you to your conclusions.

$$\mathbf{r_1} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$\mathbf{r_2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{r_3} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t_3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$\mathbf{r_4} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t_4 \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix}$$

$$\mathbf{r_5} = \begin{pmatrix} 7 \\ 3 \\ -4 \end{pmatrix} + t_5 \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$$

$$\mathbf{r_6} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + t_6 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

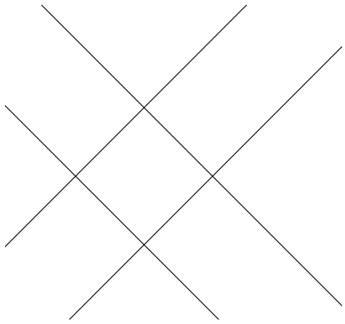
Worksheet E: Vector equation of a line

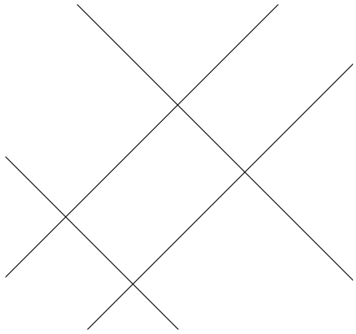
$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \mathbf{t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

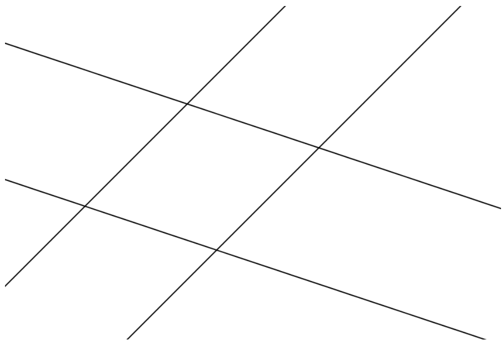
Worksheet F: Making quadrilaterals

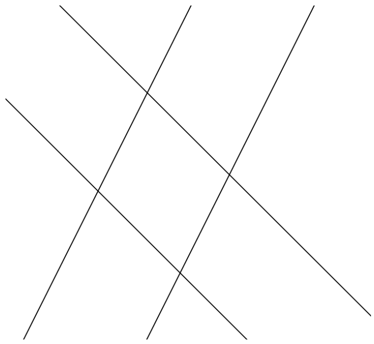
How can we create each of the following enclosed quadrilaterals from the vector equations that described the lines making the enclosed square?
For each quadrilateral, explain:

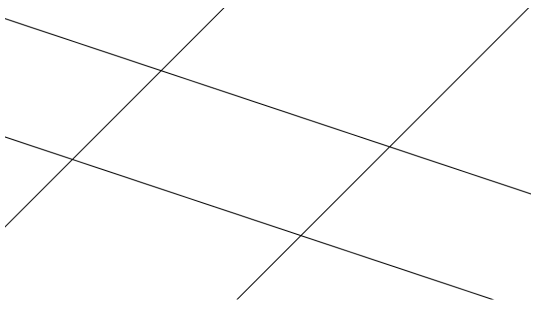
- how you changed your vector equations
- how you knew what the effect of the change was going to produce
- how you can verify that the shape you have created is the quadrilateral you require.

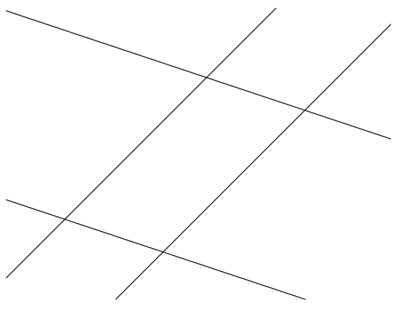




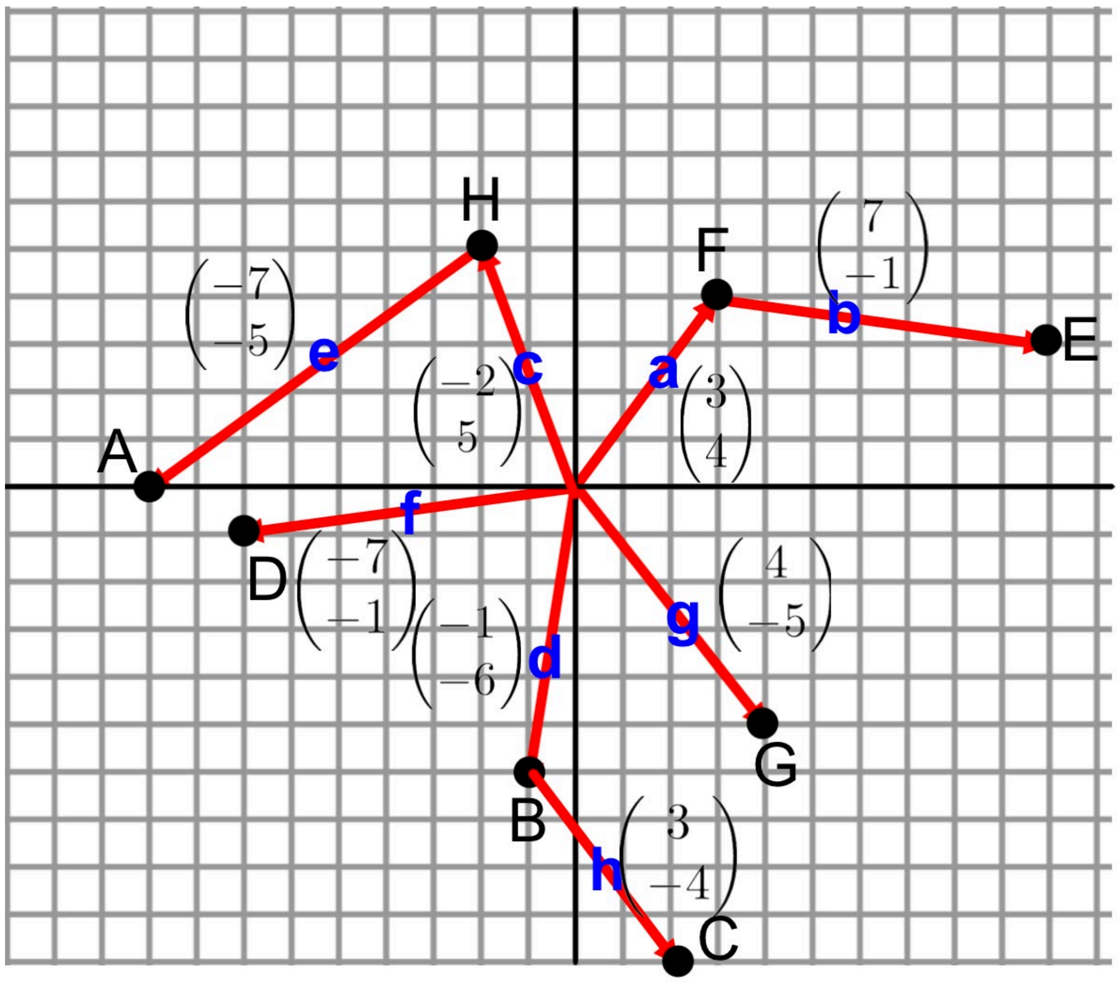








Worksheet A: Answers



Movement between points

1) $\overrightarrow{OA} = \begin{pmatrix} -9 \\ 0 \end{pmatrix}$ = **c** + **e**

2) $\overrightarrow{OE} = \begin{pmatrix} 10 \\ 3 \end{pmatrix}$ = **a** + **b**

3) $\overrightarrow{OC} = \begin{pmatrix} 2 \\ -10 \end{pmatrix}$ = **d** + **h**

4) $\overrightarrow{AE} = \begin{pmatrix} 19 \\ 3 \end{pmatrix}$ = **a** + **b** – **e** – **c**

5) $\overrightarrow{CA} = \begin{pmatrix} -11 \\ 10 \end{pmatrix}$ = **c** + **e** – **h** – **d**

6) $\overrightarrow{EC} = \begin{pmatrix} -8 \\ -13 \end{pmatrix}$ = **h** + **d** – **a** – **b**

Worksheet B: Answers

Part II

Using the diagram and $\overrightarrow{AF}$ = **a** and $\overrightarrow{BC}$ = **b**.

1) $\overrightarrow{AE}$ = $\overrightarrow{AF}$ + $\overrightarrow{FE}$ = **a** + $\overrightarrow{BC}$ = **a** + **b** and $\overrightarrow{DB}$ = $\overrightarrow{DC}$ + $\overrightarrow{CB}$ = $-\overrightarrow{CD}$ $-\overrightarrow{BC}$= $-\overrightarrow{FE}$ $-\overrightarrow{AF}$ =**- b - a**.
Hence, $\overrightarrow{AE} = k\,\overrightarrow{DB}$ where k=-1 so they are parallel.

2) $\overrightarrow{AM} = \overrightarrow{AO} + \overrightarrow{OM} = \overrightarrow{AO} + \frac{1}{2}\overrightarrow{OE} = \overrightarrow{FE} + \frac{1}{2}\overrightarrow{AF}$ = **b** + ½ **a**.

3) $\overrightarrow{MP} = \overrightarrow{ME} + \overrightarrow{EP} = \frac{1}{2}\overrightarrow{OE} + \frac{1}{2}\overrightarrow{EC} = \frac{1}{2}\overrightarrow{AF} + \frac{1}{2}(\overrightarrow{EB} + \overrightarrow{BC}) = \frac{1}{2}\overrightarrow{AF} + \frac{1}{2}(-2\overrightarrow{AF} + \overrightarrow{BC})$
= ½ **a** + ½ (-2**a** + **b**) = ½ **b** – ½ **a**.

4) $\overrightarrow{BN} = \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DN} = \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DN} = \overrightarrow{BC} + \overrightarrow{CD} + \frac{3}{5}\overrightarrow{DE} = \overrightarrow{BC} + \overrightarrow{CD} + \frac{3}{5}(\overrightarrow{DO} + \overrightarrow{OE})$
= **a** + **b** +3/5 (-**b** + **a**) = 8/5 **a** + 2/5 **b**

Worksheet C: Answers

Below are example answers. Check that your answers are equivalent.

1)
$$\mathbf{r} = t\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

2)
$$\mathbf{r} = \begin{pmatrix} -1 \\ -6 \end{pmatrix} + t\begin{pmatrix} 6 \\ -5 \end{pmatrix}$$

3)
$$\mathbf{r} = \begin{pmatrix} 10 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

4)
$$\mathbf{r} = \begin{pmatrix} 2 \\ -10 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

5)
$$\mathbf{r} = \begin{pmatrix} 2 \\ -10 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

6)
$$\mathbf{r} = \begin{pmatrix} -7 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

7)
$$\mathbf{r} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

8)
$$\mathbf{r} = \begin{pmatrix} -9 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5 \\ -4 \end{pmatrix}$$

Worksheet D: Answers

1) Equations $\mathbf{r}_1$ and $\mathbf{r}_4$ are parallel. We can tell this by looking at their direction vectors. They are parallel vectors, $\begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix} = -2 \times \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$.

We should check to make sure that they are not the same line. We can do this by checking the position vector of one of the lines. We can see that this position vector is not a position vector for a point on the other line, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$. The *y*-component would need the parameter to be $t_1 = -1$. However, the *z*-component would equal 5 and not -1.

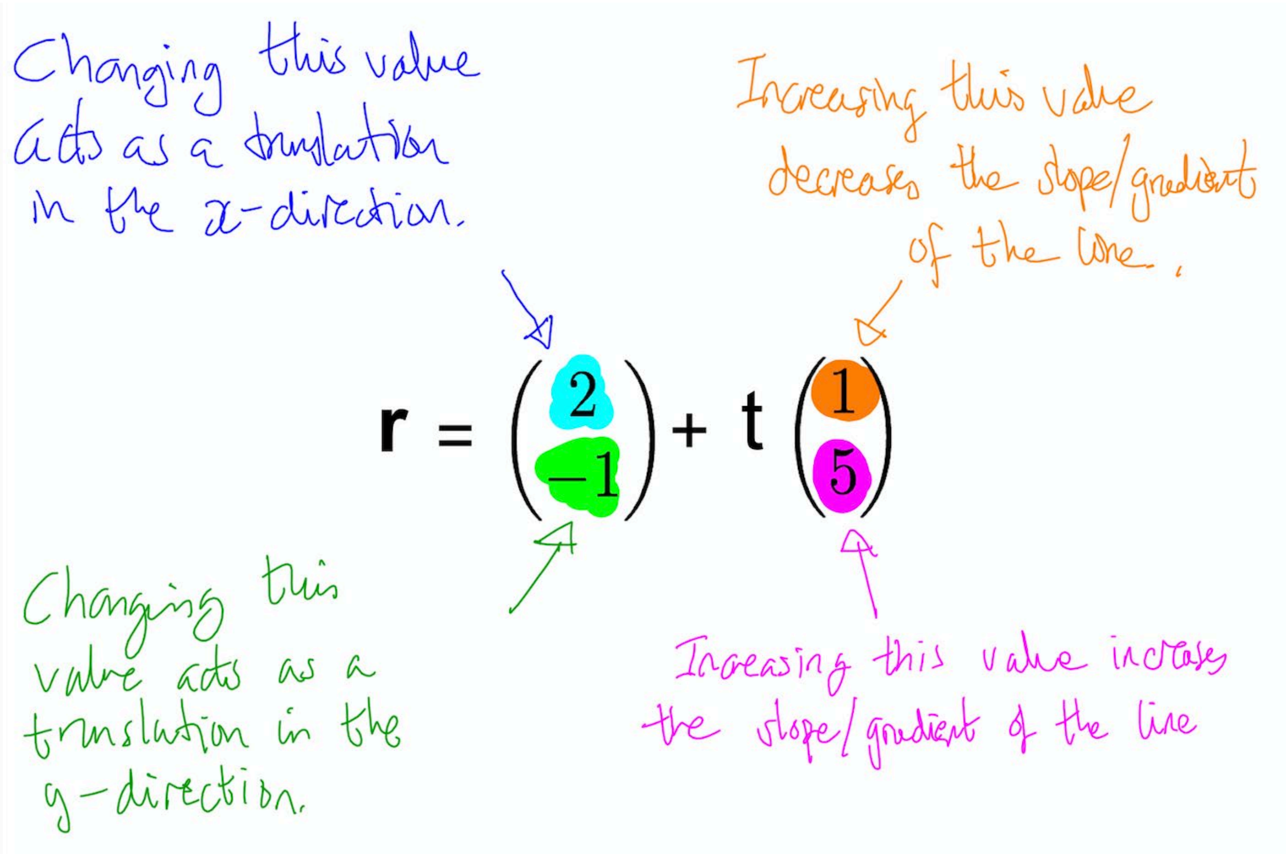
Equations $\mathbf{r}_1$ and $\mathbf{r}_5$ are parallel, as equations $\mathbf{r}_1$ and $\mathbf{r}_5$ define the same line.

2) Equations $\mathbf{r}_1$ and $\mathbf{r}_5$ define the same line. This is because their direction vectors are parallel, $\begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} = -1 \times \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$, and the position vector $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ is a position vector of a point on line $\mathbf{r}_5$ as $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -4 \end{pmatrix} - 2 \times \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$.

3) Equations $\mathbf{r}_1$ and $\mathbf{r}_2$ define a pair of lines which will intersect at the point (1,0,5). Equations $\mathbf{r}_1$ and $\mathbf{r}_3$ define a pair of lines which will intersect at the point (3,1,2). Equations $\mathbf{r}_2$ and $\mathbf{r}_4$ define a pair of lines which will intersect at the point (1,0,-1).

4) Equations $\mathbf{r}_2$ and $\mathbf{r}_6$ define lines that are parallel to the coordinate axes. $\mathbf{r}_2$ is parallel to the *z*-axis as the direction vector is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{r}_6$ is parallel to the *x*-axis as the direction vector is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Worksheet E: Answers



Worksheet F: Answers

The below answers are only examples of what you could have chosen.
Original equations for enclosed square:

$$\mathbf{r_1} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{r_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\mathbf{r_3} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} + t_3 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\mathbf{r_4} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

If we translate one of the lines by translating the position vector of the point that is used in the construction of the equation above then the new enclosed area will be a rectangle. This is because the parallel property of the lines is unaffected by changing the position vector. However, the points of intersection will change between one line and two others. For example we could translate the first line by the vector $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ to give:

$$\mathbf{r_1} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{r_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\mathbf{r_3} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} + t_3 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\mathbf{r_4} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

If we change the direction vectors of one pair of the parallel sides, then the perpendicular property will be altered and we can create a rhombus as the enclosed area:

$$\mathbf{r_1} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{r_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\mathbf{r_3} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} + t_3 \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\mathbf{r_4} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t_4 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

If we then use the same idea we used to move from the square to the rectangle, to move from the rhombus to the parallelogram we could have the equations:

$$\mathbf{r_1} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{r_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\mathbf{r_3} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} + t_3 \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\mathbf{r_4} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t_4 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

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