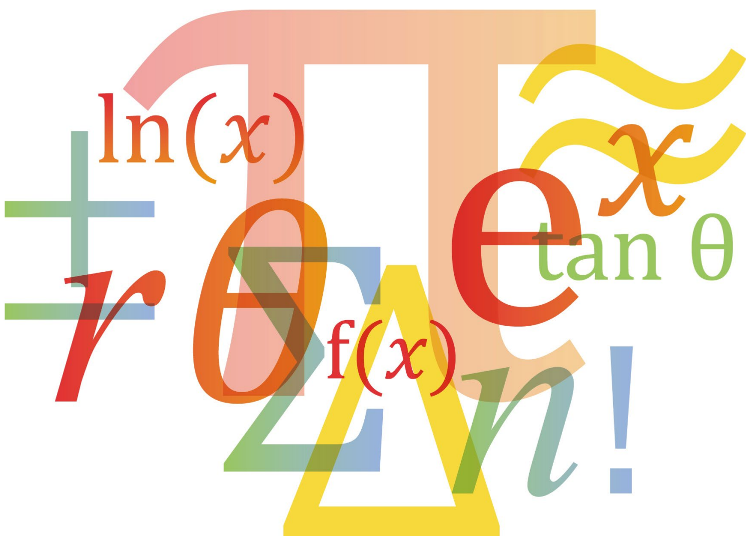
Cambridge Assessment International Education

Teaching Pack

1.3 Coordinate Geometry

Cambridge International AS & A Level Mathematics 9709



Version 1

Cambridge Pathway

In order to help us develop the highest quality resources, we are undertaking a continuous programme of review; not only to measure the success of our resources but also to highlight areas for improvement and to identify new development needs.

We invite you to complete our survey by visiting the website below. Your comments on the quality and relevance of our resources are very important to us.

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Introduction

This pack will help you to develop your learners' skills in mathematical thinking and mathematical communication, which are essential for success at AS & A Level and in further education.

Mathematical thinking and communication will be developed by focusing on:

1. Conceptual understanding – the 'why' behind the 'what'
2. Strategic competence – forming and solving problems
3. Adaptive reasoning – explanations, justifications and deductive reasoning

Throughout all activities, learners will also develop:

- procedural fluency – know when, how and which rules to use
- positive disposition – believe maths can be learned, applied and is useful
- their skills in writing mathematically – writing working & proofs

These link to the course Assessment Objectives (AOs) which you can find in detail in the syllabus:

A01 Knowledge and understanding

A02 Application and communication

Each *Teaching Pack* contains one or more lesson plans and associated resources, complete with a section of preparation and reflection.

Each lesson is designed to be an hour long but you should adjust the timings to suit the lesson length available to you and the needs of your learners.

Important note
Our *Teaching Packs* have been written by **classroom teachers** to help you deliver topics and skills that can be challenging. Use these materials to supplement your teaching and engage your learners. You can also use them to help you create lesson plans for other topics.

This content is designed to give you and your learners the chance to explore a more active way of engaging with mathematics that encourages independent thinking and a deeper conceptual understanding. It is not intended as specific practice for the examination papers.

The *Teaching Packs* are designed to provide you with some example lessons of how you might deliver content. You should adapt them as appropriate for your learners and your centre. A single pack will only contain at most five lessons, it will **not** cover a whole topic. You should use the lesson plans and advice provided in this pack to help you plan the remaining lessons of the topic yourself.

Lesson preparation

This *Teaching Pack* will cover the following syllabus content:

Candidate should be able to:	Notes and examples
• find the equation of a straight line given sufficient information	e.g. given two points, or one point and the gradient.
• interpret and use any of the forms $y = mx + c$, $y - y_1 = m(x - x_1)$, $ax + by + c = 0$ in solving problems	Including calculations of distances, gradients, midpoints, points of intersection and use of the relationship between the gradients of parallel and perpendicular lines.
• understand that the equation $(x - a)^2 + (y - b)^2 = r^2$ represents the circle with centre (a, b) and radius r	Including use of the expanded form $x^2 + y^2 + 2gx + 2fy + c = 0$.

The remaining two bullet points for topic 1.3 Coordinate geometry (shown below) are **not** covered in this *Teaching Pack*. You will need to write your **own** lesson plans to cover these items.

Candidate should be able to:	Notes and examples
• use algebraic methods to solve problems involving lines and circles	Including use of elementary geometrical properties of circles, e.g. tangent perpendicular to radius, angle in a semicircle, symmetry. Implicit differentiation is not included.
• understand the relationship between a graph and its associated algebraic equation, and use the relationship between points of intersection of graphs and solutions of equations.	e.g. to determine the set of values of k for which the line $y = x + k$ intersects, touches or does not meet a quadratic curve.

Prior knowledge and skills

For all lessons, it is assumed that learners have already completed Cambridge IGCSE™ Mathematics 0580, or a course at an equivalent level. See the syllabus for more details of the expected prior knowledge for taking Cambridge International AS & A Level Mathematics 9709.

When planning any lesson, make a habit of always asking yourself the following questions about your learners' prior knowledge and skills:

- Do I need to re-teach this or do learners just need some practice?
- Is there an interesting activity that will efficiently achieve this?

Learners will need time to revise the skills from their Cambridge IGCSE (or equivalent) coordinate geometry, in order to re-build their fluency. Even if learners mastered this topic at the time of taking the IGCSE course, it is best not to assume that they are still fluent in this topic, as this can lead to learners struggling to solve problems at AS & A Level, and believing the topic to be more complex than it actually is.

Specifically, learners should be able to:

- rearrange simple formulae, so they can see the connections between the different forms of the equations of straight lines. Practising rearranging simple formulae at the beginning of this topic in order to build fluency will save time during the main parts of the lesson later, e.g. $y = \frac{3}{2}x + 4$ is the same as $2y – 3x – 8 = 0$
- substitute negative numbers into expressions e.g. $x = –2$ into $–3x + 6$

Key learning objectives

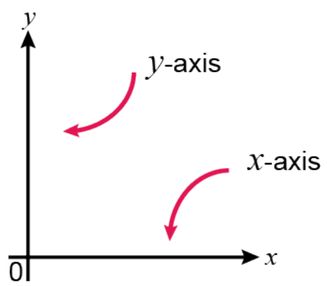
The following list represents the main underlying concepts that you should make sure your learners have understood by the end of this topic:

- A straight line is uniquely defined given any two coordinates.
- Any straight line can be described by an equation made up of an x term, a y term and a constant (any of these can be zero).
- The equation of a straight line can be expressed in a variety of forms, most commonly: $y = mx + c$; $ax + by + c = 0$ and $y – y_1 = m(x – x_1)$
- Any circle can be described by an equation made up of a term in x^2, a term in y^2, a term in x, a term in y and a constant.
- The equation of a circle can be expressed in a variety of forms, most commonly: $(x – a)^2 + (y – b)^2 = r^2$ and $x^2 + y^2 + 2gx + 2fy + c = 0$

Key terminology and notation

Your learners will need to be confident with the following terminology and notation:

***x*-axis and *y*-axis**



axes plural of axis

bisect cut something into two equal parts

Cartesian coordinates refer to the location of a point on a pair of axes (x, y)

coefficient expression in front of a variable, e.g. in $3x$, the coefficient of x is 3

equidistant the same distance from

midpoint the half-way point between two points

parallel lines lines are parallel if they are always the same distance apart (equidistant) and will never meet

perpendicular lines lines at 90° to each other are perpendicular; they are also known as ‘normal lines’

reciprocal is the quantity obtained by dividing 1 by a given quantity, e.g. the reciprocal of t is $\frac{1}{t}$

Insights video

There is an Insights video linked to this *Teaching Pack*:

- 1.3 Coordinate Geometry – watch this video which will show you how to help your learners better understand the connections between graphs and algebra.

Teacher tutorials

There are *three* tutorials linked to this *Teaching Pack*:

- **Equations of a straight line** – review this tutorial before teaching Lesson plan 1. This will demonstrate how to encourage learners to discuss what they know about coordinate geometry from IGCSE and how it forms the foundation for A-level work. This can be continued to explain the three main ways of writing an equation for a straight line using two points, which is supported by the Teacher Tutorial slides.
  The Lesson slides *Equations of a straight line* can be shown to learners in the lesson.
- **Using midpoints and distances** – review this tutorial before teaching Lesson plan 3. This will encourage learners to discuss what they already know in order to reinforce it. It will also encourage learners to visualise the situation and take an average for the midpoint, and to apply Pythagoras for the distance, which should serve to demystify the formulae they have in textbooks.
  The Lesson slides *Midpoints and distances* can be shown to learners in the lesson.
- **Equation of a circle**
  Review this tutorial before teaching Lesson plan 4. This will show you how to extend learners' knowledge of a circle centred (0,0) to equations of circles with centre (a, b), including the expanded form of the equation.
  The Lesson slides Equation of a circle can be shown to learners in the lesson.

Lesson progression

Lesson 1…Using two coordinates to find different algebraic forms for the equation of a straight line:
$y = mx + c$; $y - y_1 = m(x - x_1)$; $ax + by + c = 0$

Lesson 2… Using the gradients of equations of the lines to decide whether pairs of straight lines are parallel, perpendicular of neither

Lesson 3… Using the midpoint of two coordinates and the distance between two coordinates

Lesson 4… Using the equation of a circle to find the centre and the radius in both forms:
$(x - a)^2 + (y - b)^2 = r^2$ and $x^2 + y^2 + 2gx + 2fy + c = 0$

Lesson 5… Using coordinate geometry information to solve problems involving circles

Going forward

This topic underpins the topic of tangents and normals in the context of differentiation in topics 1.7, 2.4 and 3.4.

Lesson 1: Equations of a straight line

Preparation	• Review the Teacher tutorial *Equations of a straight line*
Resources	• Lesson slides – *Equations of a straight line* • Paper, Mini whiteboards or other writing materials • Worksheet A: *Straight line race* one between two (or one each if they want to keep them) • Squared paper (for support) • Graph sketching software or calculator (optional)
Learning objectives	By the end of the lesson: • ***all*** learners should be able to find the equation of a line in the form $y = mx + c$ given two coordinates or a gradient and one coordinate • **most** learners should also be able to identify the key features of a line given by equations in the form $ax + by + c = 0$ and $y - y_1 = m(x - x_1)$, converting from one form to another • **some** learners will also be able to identify which form of a line is the best to use in a given situation

Dependencies

Learners need to know how to find the gradient from two pairs of coordinates. (This is revised in Lesson 1.)
They should be familiar with equations in the form $y = mx + c$, where m is the gradient and c is the y-intercept.
They should know how to find the point of intersection of two lines (This is practised in Lesson 1 through a simple Arithmagon.)

Common misconceptions

Misconception	Problems this can cause	An example way to resolve the misconception
That the coefficient of x is always the gradient and the y intercept is always the number on its own.	If learners believe this then they will not be able to successfully solve problems.	Give an example for learners to draw, e.g. $2y + 3x - 12 = 0$ and ask them to **work out** the gradient and y-intercept from the graph. They compare their calculated values to the equation and see for themselves that the gradient is not 3 and the y intercept is not –12. A graph-drawing package will help speed up this process.
Being inconsistent with finding the gradient from two pairs of coordinates e.g. (2,3) and (4,1) they calculate 4 – 2 and 3 – 1 instead of 4 – 2 and 1 – 3.	Their gradient will be wrong and the equations will not be correct.	Demonstrate both and show the negative effect on the gradient.

Timings	Activity
	Starter/Introduction
5 min	**Warm-up activity: To build mathematical language and encourage discussion** Lesson slide 2 Draw one coordinate on a coordinate grid; the numbers should be simple. Discuss in pairs: 'What can you tell me about this coordinate?' Use this question to gauge learner's prior knowledge. It is important that if a couple of learners give complex suggestions you do not assume they all have that fluency. Learners record their findings on their mini whiteboards. Possible answers are: The x coordinate is 2; the y coordinate is 3 (that is a good start, it is surprising how many A level students can get them the wrong way round) Encourage all findings but focus on the equations of some lines that the coordinate lies on, e.g. It lies on the line x=2 and the line y=3 Reinforce that $x = 2$ and $y=3$ satisfy each of the equations – 'they make it true' e.g. It lies on the line $y=x+1$; $y=2x-1$ etc. Ask: 'How do you know?' Ask: 'Does it lie on the line y=4*x*+2?' 'No.' 'Convince me.' Ask: 'How many lines does it lie on?' Answer: An infinite number.
5 min	Lesson slide 3 **Individually**: **To build fluency in simultaneous equations** Learners can quickly fill in the Arithmagon to find the three coordinates that make it true. Give learners 5 minutes to find the three coordinates to place in the circles that make this true. Show the graphs of the three equations to reinforce the connection. This can be adapted for any equations if learners need further practice. **Challenge:** Learners could make them up for each other and could be challenged to make integer coordinates.
5 min	Lesson slides 4 to 7 **Individually**: **To check that learners can remember how to calculate the gradient of a line between two given points.** Use mini whiteboards (or A3 sheet of paper) and slides 4 to 7. **Support:** For less able learners, you might need to provide squared paper so they can see the gap in *y* and in *x*. Use the prompts and questions on the Teacher tutorial to encourage learners to recall their IGCSE knowledge about coordinates and straight lines and share their ideas.

<table>
<tr><th>Timings</th><th>Activity</th></tr>
<tr><td>5 min</td><td>
In pairs: To build fluency in gradient calculation, as errors in the gradient are often the root cause of errors in coordinate geometry questions
Use mini whiteboards. Ask learners to write two coordinates including negatives on their board. Without their partner seeing, they calculate and write the gradient on the back of the board.
Learners swap boards with their partner and race to see who calculates the gradient first. The competitive nature will encourage them to choose particularly challenging coordinates. If they disagree, they should seek judgement from a third learner or you.
Make sure that learners are not just following the rule $gradient = \frac{y_2 - y_1}{x_2 - x_1}$ but can also visualise the relative position of the coordinates, as a safety check that they are not getting a negative gradient when they should have a positive one.
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<tr><td>20 min

15 min</td><td>
Main lesson
Main objective: To calculate the equation of a straight line from two points
Lesson slides 8 and 9
Demonstrate that all you need for a unique equation of a straight line is to know two points, e.g. (2, 3) and (5,12).
Lesson slide 10
Demonstrate how to find an equation from these same two points in the forms:
$y = mx + c$
$y - y_1 = m(x - x_1)$
$ax + by + c = 0$
Explain that the last of these forms of the equation does not have a gradient ‘m’ so it is better to find one of the other forms and rearrange if this is the required format.
By using the same line each time, it should reinforce that they are just different forms of the same line with nothing mysterious.
Slide 10 is a good summary for learners to put in their notes. Confirm understanding by using the practice question on slide 11.
Worksheet A: Straight line race
In pairs: Use mini whiteboards for working out.
Ask learners to fill in the table in pairs. The race aspect enables them to focus on which format is more efficient.
This will reinforce finding the gradient and also allow learners to see their preferred way of working out the equation.
Challenge: Ask learners how they can determine if three points are colinear. This will require them to think about what colinear means as well as how they could determine this.
Support: Encourage learners to try sketching given scenarios on a grid. Ask questions such as: ‘Can you write that in a different way? Can you rearrange it?’
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5 min	**Plenary** Worksheet A: *Answers* Ask learners for their contributions of answers to the questions on Worksheet A. Ask: 'Which form did you find easiest to work out?'

Reflection	Reflect on your lesson; use the **Lesson reflection** notes to help you.

Lesson 2: Parallel, perpendicular or neither

Resources	• Paper, Mini whiteboards or other writing materials • Lesson slides: *Parallel, perpendicular or neither* • Worksheet B: *Parallel and perpendicular lines notes* one each • Worksheet C: *Parallel, perpendicular or neither* display on the board or one between two • Worksheet D: *Problems with straight lines* one between two • Worksheet E *Lots of Lines* per pair

Learning objectives	By the end of the lesson: • ***all*** learners should be able to identify perpendicular gradients in equations in the form $y = mx + c$ • **most** learners should be able to identify perpendicular gradients in equations given in any form • **some** learners will be able to explain the reasons why lines are perpendicular

Timings	Activity
5 min	**Starter/Introduction** Check learners' understanding of gradients and perpendicular gradients by asking a series of open questions. Learners respond on a mini whiteboard so that you and the rest of the class can see. Example questions include: • Show me an example of an equation of a line that has a gradient of 4. • Show me the equations of two lines that are parallel. • Show me an example of an equation of a line that is perpendicular to $y = 3x – 2$. • Show me the equations of two lines that are perpendicular. Encourage learners to view each other's work and question any that they do not agree with. This improves communication and the articulation of the key vocabulary. Typical learner responses to the first bullet might be to show '$y = 4x + c$' (they might have replaced c with a value). If some show $y = mx + 4$ they have misunderstood the gradient / intercept form of an equation and you need to refer them back to $y = mx + c$ where m is the gradient, c is the y-intercept (crosses the y-axis when $x = 0$). More challenging responses could include $2y – 8x = 6$. Encourage and praise this type of response as the learner has grasped a different form of the equation, using their knowledge from Lesson 1. **Support:** If learners have struggled to remember the properties of parallel and perpendicular lines from IGCSE then give them Worksheet B: *Parallel and perpendicular lines notes* for their notes.

Timings	Activity
10 min 30 min	**Main lesson** Worksheet C: Parallel, perpendicular or neither (one between two) **In pairs:** Learners need to decide whether each pair of equations is parallel, perpendicular or neither and be able to convince someone else i.e. justify their decision. Use 'Think-pair-share' but the pair will share with another pair, rather than the whole class (see *How to engage your learners* guide for details of this technique). Each time the equations are discussed, one learner is chosen to provide evidence to support their decision and the listener has an opportunity to challenge it if they want to. **Main objective: To build accuracy in mathematical writing** The next activity involves individual thinking time followed by class discussion. The series of problems are available on Worksheet D: *Problems with straight lines* so that you can display them as appropriate (possible solutions are supplied in Worksheet D: *Answers* to discuss how to set out a solution in an examination). Display the following question for the whole class to see: ***Find the equation of the line parallel to y = 2x + 4 which passes through (4, 9).*** Give learners about 3 minutes to answer the question individually. Invite a learner to come up to the front to solve the problem – ask them to talk through their working as they go. Invite other learners to ask questions or provide comments on the solution. Emphasise that it doesn't matter if the answer is wrong, it's important to have a go as we learn from our mistakes. If the learner hasn't got it quite right, invite another to amend the solution, and so on, until the correct answer is reached. Ask if other learners got the same result but used a different method – ask them to demonstrate their method. Discuss which they think is the most efficient and why. The discussions should lead learners to realise that the answer is any line with the equation $y = 2x + c$, but only one of the family of lines goes through (4, 9). They should also realise that the value of c in the equation has nothing to do with the x-coordinate in the point. Now display the following question for the whole class to see and repeat the answer and discussion process as before: ***Find the equation of the line perpendicular to y = 2x − 3 which passes through (−2, 5).*** And then finally, repeat for: ***A line is perpendicular to the line*** $2y - x - 8 = 0$, ***and passes through the coordinate (5,-7). Find the equation of the line.*** With Question 3 it is important that learners show that they can use either $y = mx + c$ or the method given in the answers. Learners need to find their own preferred method.
15 min	**Plenary** Worksheet E: Lots of lines **In pairs:** Give each pair a copy of: Properties and Card set – Equations. Ask learners to cut out the cards on the paper and sort them according to the properties in the table. Encourage them to discuss and justify their decisions using the correct mathematical terminology, before placing each card in the table. If you want them to

Timings	Activity
	present their findings, or you want to keep them for another lesson, ask them to glue the cards in place. Learners can use a mini whiteboard or scrap paper if they want to work anything out. Towards the end of the activity, allow each pair to check their work on a graphical calculator. The equations have been chosen to highlight the possible misconception that the coefficient of x is always the gradient and the y intercept is the number at the end. Learners should find that there are two equations in each column of the table, leaving two left over for which they need to define the property that links them. The missing property is ‘passing through the point of intersection of the two lines’. They are asked to name the final property so that they cannot match the last two properties by default (i.e. there are only two cards left so they must fit into the last property). **Challenge:** Give learners some blank cards to write another equation for each property and ask them to write a justification for why they match. Discuss the sorting activity as a class by asking questions such as: Why were these two not parallel? How do you know that these are perpendicular? These two equations have both got a ‘5’ at the end. Why do they not have the same y-intercept? Example learner responses might include them looking for ‘same gradient’ in their answer, looking for the negative reciprocal in their answer; looking for the fact that they are not both in the form $y = mx + c$.

Reflection	Reflect on your lesson; use the **Lesson reflection** notes to help you.

Lesson 3: Problems with straight lines

Preparation	• Review the Teacher tutorial *Using midpoints and distances*.
Resources	• Lesson slides: *Using midpoints and distances* • Worksheet F: *Problem-solving anagram* one each • Worksheet G: *Same points, different answer* one each • Paper, Mini whiteboards or other writing materials
Learning objectives	By the end of the lesson: • ***all*** learners should be able to find the distance between two coordinates and to find the midpoint of two coordinates • **most** learners should be able to solve problems involving distances and midpoints together with parallel and perpendicular lines • **some** learners will be able to solve complex problems involving distances, midpoints together with parallel and perpendicular lines by identifying the most efficient method to use

Timings	Activity
15 min	**Starter/Introduction** **Main objectives:** • **Find the distance between two points** • **Find the midpoint of two points** Lesson slide 2 Put two coordinates on a plain whiteboard e.g. (2,3) and (5,9). (Slide 2) Ask: 'How far is it between them? Learners should see the points as vertices of a right-angled triangle. Visualising this allows the use of Pythagoras' theorem to follow naturally. Use Slide 3 to confirm the learners solution. Extend the difficulty by introducing negative coordinates (-3, 6) and (5,15). It is vital that the distance between the *x* values is consistent with the difference between the *y* values. Some learners may struggle subtracting negative numbers… the distance between -3 and 5 is not 2. Need 5 – -3 = 8 Similarly ask: 'What is the coordinate that is midway between the two points e.g. (2,3) and (5,9)? Learners should see it as the average so add the *x* coordinates together and divide by 2, add the *y* coordinates together and divide by 2. Extend the difficulty by using negative coordinates (-3, 6) and (5,15) It is vital the the negative numbers are used correctly when calculating the midpoint. Some learners may struggle adding negative numbers or using their calculator (-) symbol correctly. Solutions are provided on Slides 5 and 6 to support your learners understanding and appropriate communication. Extend to the general rules used to calculate the distance between two points and their midpoint (Slide 4 - this can also be given as notes for learners to revise from.)

	Build fluency by learners answering quick-fire quiz questions. Invite a learner to write two points on the board and ask for : • gradient • midpoint • distance between them Learners can race to give the answer in quiz form. Lesson slides 7 and 8 As a final challenge, use coordinates (a,b) and (3a,5b) and ask the learners to individually work out the same information (Slide 7). Ask a different learner to present each result. Slide 8 can be used to confirm the solutions.
	**Main lesson** Worksheet F: *Problem-solving anagram* Lesson slide 9 Handout the worksheet, or put it up on the board. **Individually:** Learners work individually to decide whether each statement is true. They must have a reason for their decision. Once they have finished they can spend time trying to work out the anagram for the name of a famous mathematician. (The right answers have the letters 'Descartes' so students can learn a little history too.) René Descartes made an important connection between geometry and algebra, which allowed for the solving of geometrical problems by algebraic equations. Once most learners have completed the grid they can pair and share. The answers are on Worksheet F: *Answers*.
	**Plenary** Worksheet G: *Same points, different answer* Students often struggle with problems that seem familiar but are asking a different question. This sheet tackles this issue head on.

Reflection	Reflect on your lesson, use the **Lesson reflection** notes to help you.

Lesson 4: Equations of circles

Preparation	• Review the Teacher tutorial *Equation of a circle*.
Resources	• Lesson slides: *Equation of a circle* • Paper, Mini whiteboards or other writing materials • Worksheet H: *Tarsia Jigsaw* – Circles equations, radius and centre, one between two (or one each if they want to keep them) • Scissors (or pre-cut the jigsaw) • Large paper to stick the jigsaw to (if they want to keep it) • Graph sketching software or calculator (optional)
Learning objectives	By the end of the lesson: • ***all*** learners should be able to identify the centre and radius of a circle given in the form $(x-a)^2+(y-b)^2=r^2$ as (a,b) and r • **most** learners should also be able to find the centre and radius of a circle given in the form $x^2+y^2+2gx+2fy+c=0$ • **some** learners will be able to find the equation of a circle from sufficient information

Dependencies

Learners will need to have mastered completing the square, and already be familiar with the equation of a circle with centre (0, 0).

Common misconceptions

Misconception	Problems this can cause for learners	An example way to resolve the misconception
Thinking that $(x+10)^2+(y+15)^2=4$ has centre at (10, 15)	The circles will be in the wrong place.	Graph plotting software shows the circles to be centred correctly with the correct radius.
Thinking that $(x+10)^2+(y+15)^2=4$ has radius 4, i.e. forgetting to square root r^2	The circles will be too big if you forget to square root.	

Timings	Activity
20 min	**Starter/Introduction** **Main objective: Find the bracket form of the equation of a circle** Lesson slides 2 and 3 Use these slides to draw the equation out of learners from their previous knowledge of the distance between two points. On mini whiteboards learners individually write the equation of the general circle centre (*a*,*b*) radius *r*. Learners should have held up $(x-a)^2+(y-b)^2=r^2$ Ask learners to multiply out the brackets to get $x^2+y^2-2ax-2by+a^2+b^2-r^2=0$

Timings	Activity
	Ask learners 'How do we get back to the bracketed form where we can pick off the centre and radius?' This is a harder task than completing the square on just one variable and learners may need practice. The Tarsia jigsaw gives practice in this skill as well as finding the centre and radius.
20 min 10 min	**Main lesson** **Objective: To build fluency in connecting the key features of centre and radius with the equation** Worksheet H: *Tarzia puzzle* In pairs, learners complete the Tarsia Jigsaw to build fluency in matching the equation, in both forms, with the centre and radius. Learners can stick the puzzle onto larger sheets of paper to keep it if desired. Use the printout of the solution for learners to check their own **If time permits** Ask a volunteer to write up an equation of a circle in one form and ask learners: 'What is the centre?' 'What is the radius?' Ask a volunteer to write up an equation of a circle in the other form and ask learners: 'What is the centre?' 'What is the radius?' 'Which form is easier?' Reiterate how to complete the square to transfer to the 'easier form'.
10 min	**Plenary** Lesson slide 4 Use the diagram as a discussion point for what information is needed to find the equation of a circle. Prompt: 'We need the centre and the radius.' **'Does this give us the centre?'** No. 'Can we find the centre?' Yes, it is the midpoint of the ends of the diameter. **Does this give us the radius?** No 'Can we find the radius?' Yes, it is the half the distance of the diameter. Or the distance from the centre to an end point.

Reflection Reflect on your lesson; use the **Lesson reflection** notes to help you.

Lesson 5: Problems with circles

Resources

- Paper, Mini whiteboards or other writing materials
- Worksheet I: *Circle theorems* one each
- Worksheet J: *Goal-free problem* one between two
- Worksheet K: *Redundant information* to display

Learning objectives

By the end of the lesson:

- ***all*** learners should be able to solve problems with equations of the form $(x-a)^2 + (y-b)^2 = r^2$ with use of basic circle theorems
- **most** learners should be able to solve problems with equations in the expanded form $x^2 + y^2 + 2gx + 2fy + c = 0$. with use of basic circle theorems
- **some** learners will be able to switch fluently between the two forms as required by the problem and efficiently solve complex problems with the use of basic circle theorems

Timings	Activity

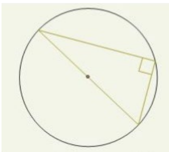
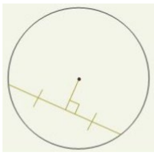
Starter/Introduction

10 min

Warm-up activity: To build mathematical language for the circle and encourage discussion of key circle theorems

Key words:
Chord, Diameter, Radius, Tangent, Bisect, Perpendicular, Right angle

Worksheet I: *Circle theorems*


Snowball activity

Use the three diagrams to build the language needed to solve many coordinate geometry circle problems. The snowball activity encourages learners to explain using efficient terminology for the diagrammatic mathematical representation.

Ask learners to discuss the three diagrams in pairs and record findings on mini whiteboard. After 1 minute ask them to share their findings with another pair, to check accuracy and build up the knowledge base.

Ask volunteers to tell the class what they know.

Worksheet I: *Answers*
Finally reveal the circle theorems for learners to make notes.

Challenge: For more able learners give them the diagrams without the right angle symbols.

Main lesson

Main objective: To encourage the concept of sketching the information: *'A picture paints a thousand calculations or can often save a thousand calculations.'*

25 min	Worksheet J: *Goal-free problem* **The points P(3, 4) and Q(5, 10) are on the circumference of the circle with centre O(10, 5)** **What can you find out from this information?** **In pairs:** Give learners Worksheet H and ask each pair to write down as much as they can using the information given. Encourage them to use diagrams. It is possible to play **'Last one standing'** – who found something that no one else did – with this type of problem to encourage learners to seek the most obscure facts. . Alternatively you can ask (Use Worksheet I: Answers): Who found the radius? $\sqrt{50}$ Who found the midpoint of the chord? M(4,7) Who found the length of the chord? $\sqrt{40}$ Who found the equation of the perpendicular bisector? $x + 3y - 25 = 0$ Who found the equation of the circle? $(x - 10)^2 + (y - 5)^2 = 50$ Who found that PQ is the same length as OM? $\sqrt{40}$ Who found the area of the triangle POQ? 20 units squared $\frac{\sqrt{40}\sqrt{40}}{2}$ because of the right angle
25 min	**Plenary** Worksheet K: *Redundant information* To further encourage learners to use diagrams and to discuss redundant information use the following question given on Worksheet I. **The points A(6,6), B(6,-2), C(-1,-1) and D(-2,2) are four points on a circle.** **Find the equation of the circle.** Ask learners if they could do without any of the information given in the question. Some suggestions are given in Worksheet K: *Answers*. To follow on from Lesson 1 and the fact that two points are necessary to uniquely identify the equation of a line, ask how many points are necessary to uniquely identify a circle? This is discussed on Worksheet K: *Answers*. If time permits learners could try two points to see how many equations they can find that go through them.

Reflection	Reflect on your lesson; use the **Lesson reflection** notes to help you.

Planning your own lessons

You now need to plan lessons to cover the following bullet points:

Candidate should be able to:	Notes and examples
• use algebraic methods to solve problems involving lines and circles	Including use of elementary geometrical properties of circles, e.g. tangent perpendicular to radius, angle in a semicircle, symmetry. Implicit differentiation is not included.
• understand the relationship between a graph and its associated algebraic equation, and use the relationship between points of intersection of graphs and solutions of equations.	e.g. to determine the set of values of k for which the line $y = x + k$ intersects, touches or does not meet a quadratic curve.

Follow the structure of the *Teaching Pack*, and use techniques from the 'How to' guides, to create your own engaging lessons to cover these bullet points. Consider what preparation you need for each lesson: what prior knowledge is needed, what are the key objectives, what are the dependencies, what common misconceptions are there, and so on.

Below, we have provided an outline of some activities and approaches you might like to try.

Lesson *x*: Solving problems with lines and circles
Common misconceptions: Not drawing a clear diagram to visualise the information given in the problem
Starter: You could try using a question without a diagram that requires the simplest of the circle theorems "a tangent meets the radius at 90 degrees" ask students to draw a diagram and "Think, pair, share to discuss which circle theorem is needed to solve the problem. For example

A circle with centre (-2,4) passes through the point A (a,1). Given that the tangent to the circle at A has gradient $\frac{5}{3}$ find the value of a.

Main: You could use Worksheet L *Building a picture* and have a class discussion about;

- the need to visualise the information given
- the need to use other known facts about a mathematical situation to formulate a solution.
- Ask "What else can we find from this information?"

Plenary: You could try using past examination style questions but just ask "What do we need to do?" rather than actually completing the solution. This builds confidence and fluency.

Lesson *y*: The relationship between a graph and its algebraic equation
Common misconceptions: Use of the x-intercept of a graph as the points where x=0 not y=0, and vice versa.
Starter: You could try using the intersection with just the x or y axes before any other lines are used. For example:

"Find the points where the circle $(x - 3)^2 + (y - 8)^2 = 80$ intersects the x-axis"

Answer: Using $(x - 3)^2 + (0 - 8)^2 = 80$ gives *(7,0) and (-1,0)*

Does it intersect the y-axis?

Answer yes, at $(0, 8+\sqrt{71})$ *and at* $(0, 8-\sqrt{71})$

Main: You could use a Graph Plotting Package such as Desmos to graph various quadratics and lines or circles and lines, ask students to find the points of intersection algebraically and then find them on the package to check.

Plenary: You could have a class discussion from the main activity about when the resulting quadratics have one, two or no real solutions and how this discriminant characteristic is useful for the solution to how many times a line meets a circle, a line meets a quadratic.

You will find some other activity suggestions in the Scheme of Work.

Lesson reflection

As soon as possible after the lesson you need to think about how well it went.

One of the key questions you should always ask yourself is:

Did all learners get to the point where they can access the next lesson? If not, what will I do?

Reflection is important so that you can plan your next lesson appropriately. If any misconceptions arose or any underlying concepts were missed, you might want to use this information to inform any adjustments you should make to the next lesson.

It is also helpful to reflect on your lesson for the next time you teach the same topic. If the timing was wrong or the activities did not fully occupy the learners this time, you might want to change some parts of the lesson next time. There is no need to re-plan a successful lesson every year, but it is always good to learn from experience and to incorporate improvements next time.

To help you reflect on your lesson, answer the most relevant questions below.

Were the lesson objectives realistic?
What did the learners learn today? Or did they learn what was intended? Why not?
Were there any common misconceptions?
What do I need to address next lesson?
What was the learning atmosphere like?
Did my planned differentiation work well?
How could I have helped the lowest achieving learners to do more?
How could I have stretched the highest achieving learners even more?
Did I stick to timings?
What changes did I make from my plan and why?

Summary evaluation

What two things went really well? (Consider both teaching and learning.)

What two things would have improved the lesson? (Consider both teaching and learning.)

What have I learned from this lesson about the class or individuals that will inform my next lesson?

Worksheets and answers

Worksheet A: Straight line race

Work in pairs to fill in the table below.

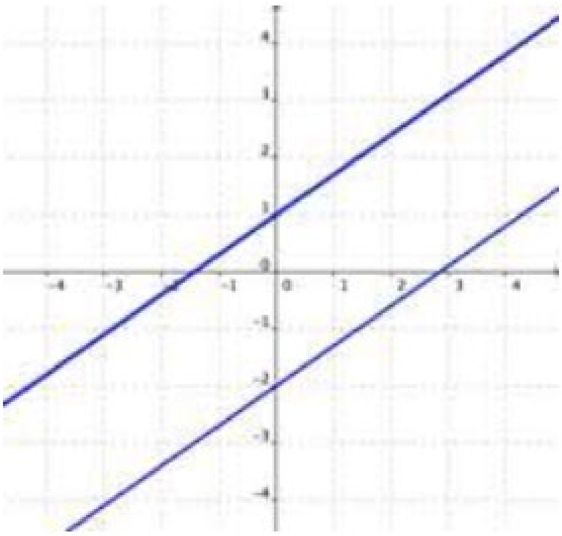
For the $y - y_1 = m(x - x_1)$ column you may have a choice of answers
For the $ax + by + c = 0$ column a, b, and c are integers.

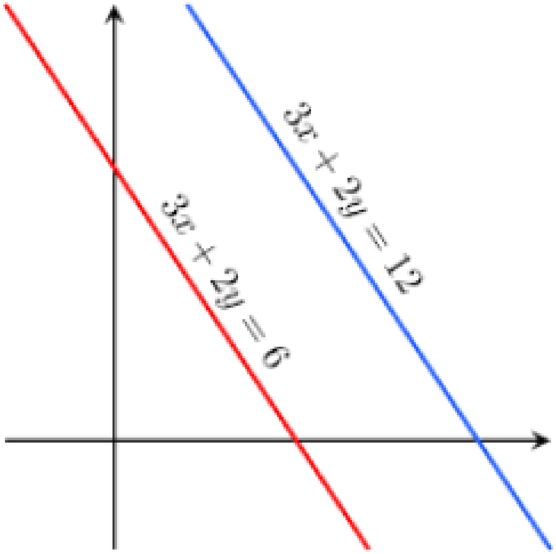
Coordinate Pair	$y=mx+c$	$y - y_1 = m(x - x_1)$	$ax + by + c = 0$
(5,2) and (3,4)			
(9,-1) and (7,2)			
(-6,1) and (4,0)			
(-12,3) and (5,7)			

Worksheet B: Parallel and perpendicular lines notes

Parallel lines

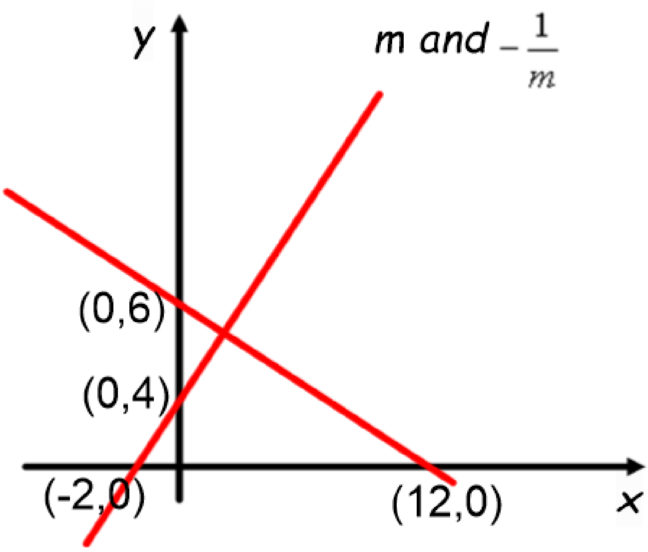
- **Parallel lines have the same gradient.**

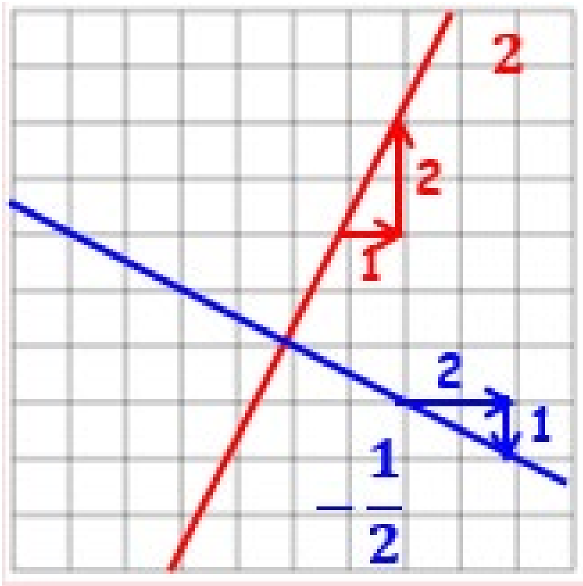




Perpendicular lines

- **If two lines are perpendicular, their gradients have a product of –1**
- If two lines are perpendicular, they are at right-angles to each other. A line perpendicular to the line with equation $y = mx + c$ has gradient $-\frac{1}{m}$.





Worksheet C: Parallel, perpendicular or neither?

Decide whether each pair of equations is:

- parallel
- perpendicular
- neither.

Justify your decision.

a $y = 2x + 3$
$y = 2x - 7$

b $y = 3x$
$2x + y - 3 = 0$

c $y = 4x - 3$
$4y + x = 2$

d $3x - y + 5 = 0$
$x + 3y = 1$

e $2x + 5y - 1 = 0$
$y = 2x + 7$

f $2x - y = 6$
$6x - 3y + 3 = 0$

Worksheet D: Problems with straight lines

Question 1

Find the equation of the line parallel to $y = 2x + 4$ which passes through (4, 9).

Question 2

Find the equation of the line perpendicular to $y = 2x - 3$ which passes through (−2, 5).

Question 3

A line is perpendicular to the line $2y - x - 8 = 0$, and passes through the point with coordinate (5, -7). Find the equation of the line.

Worksheet E: Lots of lines

Equations

$y = 4x + 4$	$4y = x + 3$
$y = 8x - 3$	$y + 4x + 6 = 0$
$3y = 2x - 8$	$y + 6x = 11$
$y + 8x = 6$	$2y + 8 = 3x$
$2y + x = 4$	$2y = 8x + 3$
$y = 6x - 4$	$y + x + 8 = 0$

Worksheet E: Lots of lines continued

Properties

These lines are perpendicular.	**These lines have the same x intercept.**	**These lines .....**
These lines are parallel.	**These lines have the same y intercept.**	**These lines both go through the point (1, 5).**

Worksheet F: Problem-solving anagram

The coordinates of A, B and C are (1,3), (-3,2) and (-2,-2)

Which of these statements are correct?

V Perpendicular bisector of B and C has a gradient of $-\frac{1}{4}$	**A** AB is perpendicular to BC	**R** The midpoint of A and B is (-1, 2.5)
C The length of AB is $\sqrt{17}$	**S** The gradient of BC is -4	**B** The midpoint of A and C is $(\frac{1}{2}, \frac{1}{2})$
T A and C are points on the line 3y = 5x + 4	**E** The gradient of AC is $\frac{5}{3}$	**D** Equation of the line through B and C is $y+4x+10=0$
H AC is perpendicular to AB	**E** Perpendicular bisector of A and C is $5y+3x-1=0$	**S** The area of triangle ABC is 8.5cm^2

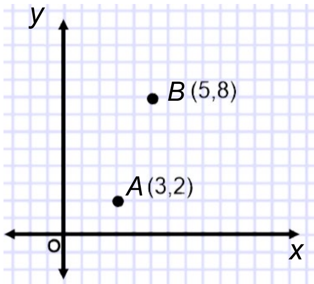
The letters from the True statements can be rearranged into which famous mathematicians name?

……………………………………..…………………………

Worksheet G: Same points, different answer

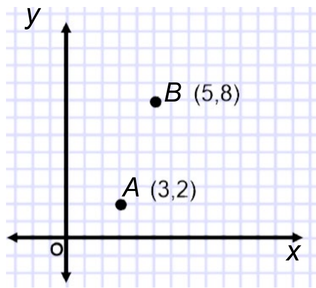
1

Work out the coordinates of the midpoint of AB.



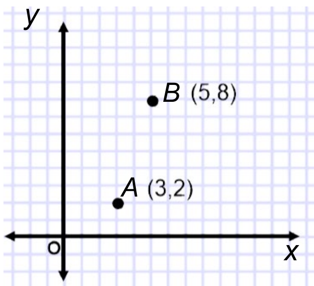
2

Find the length of the line AB.



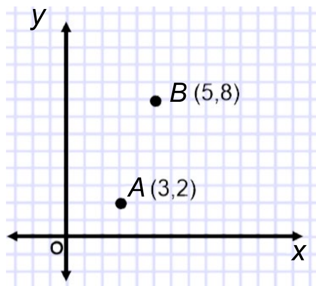
3

Work out the gradient of the line of AB.



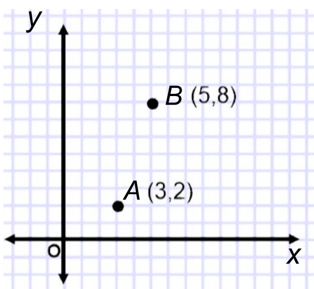
4

Work out the equation of the line passing through A and B.



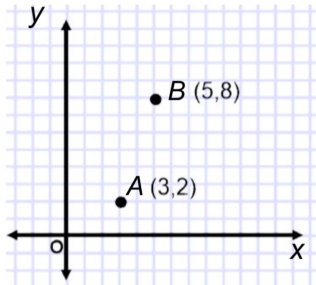
5

Work out the equation of the perpendicular bisector of AB.

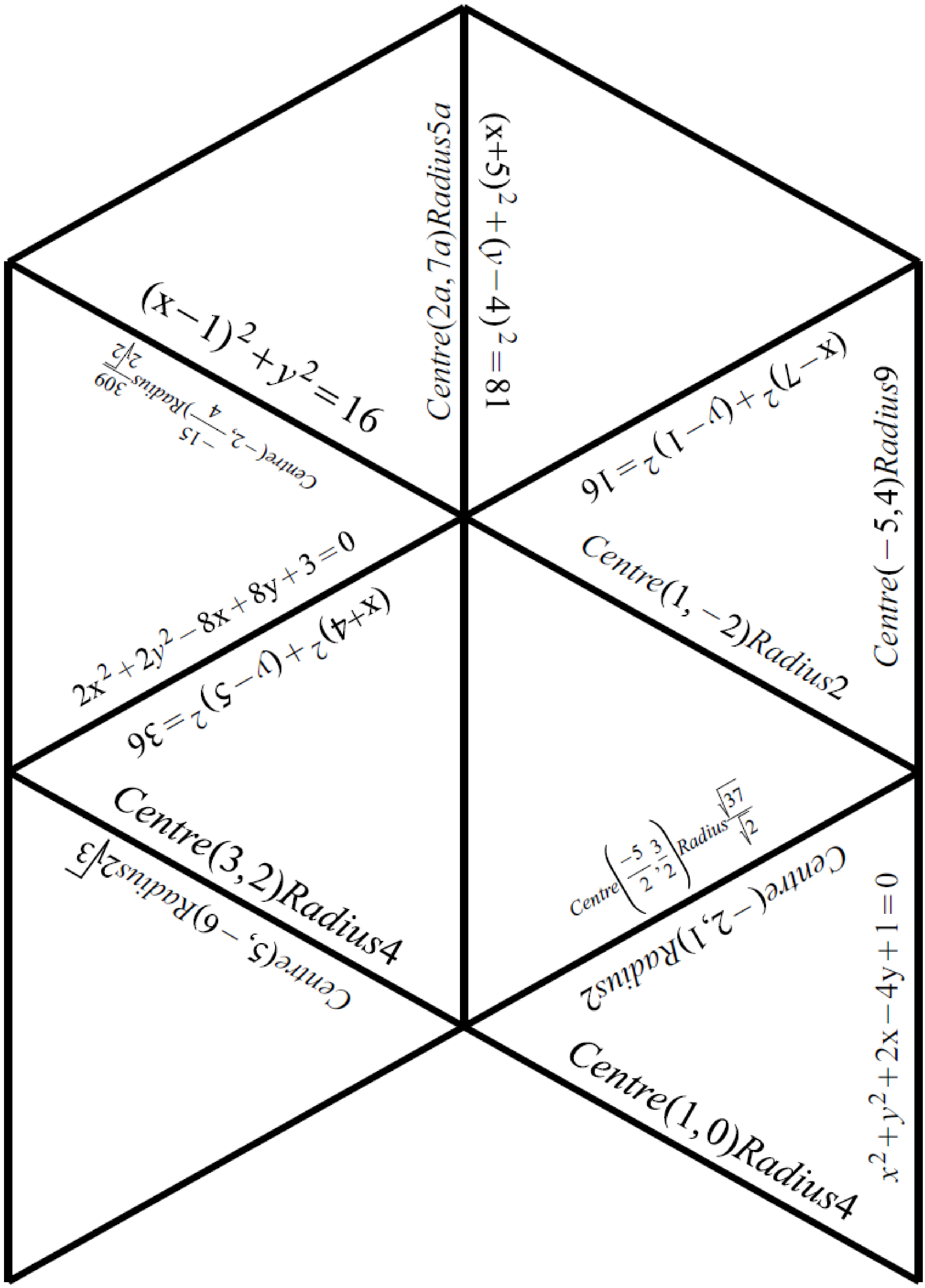


6

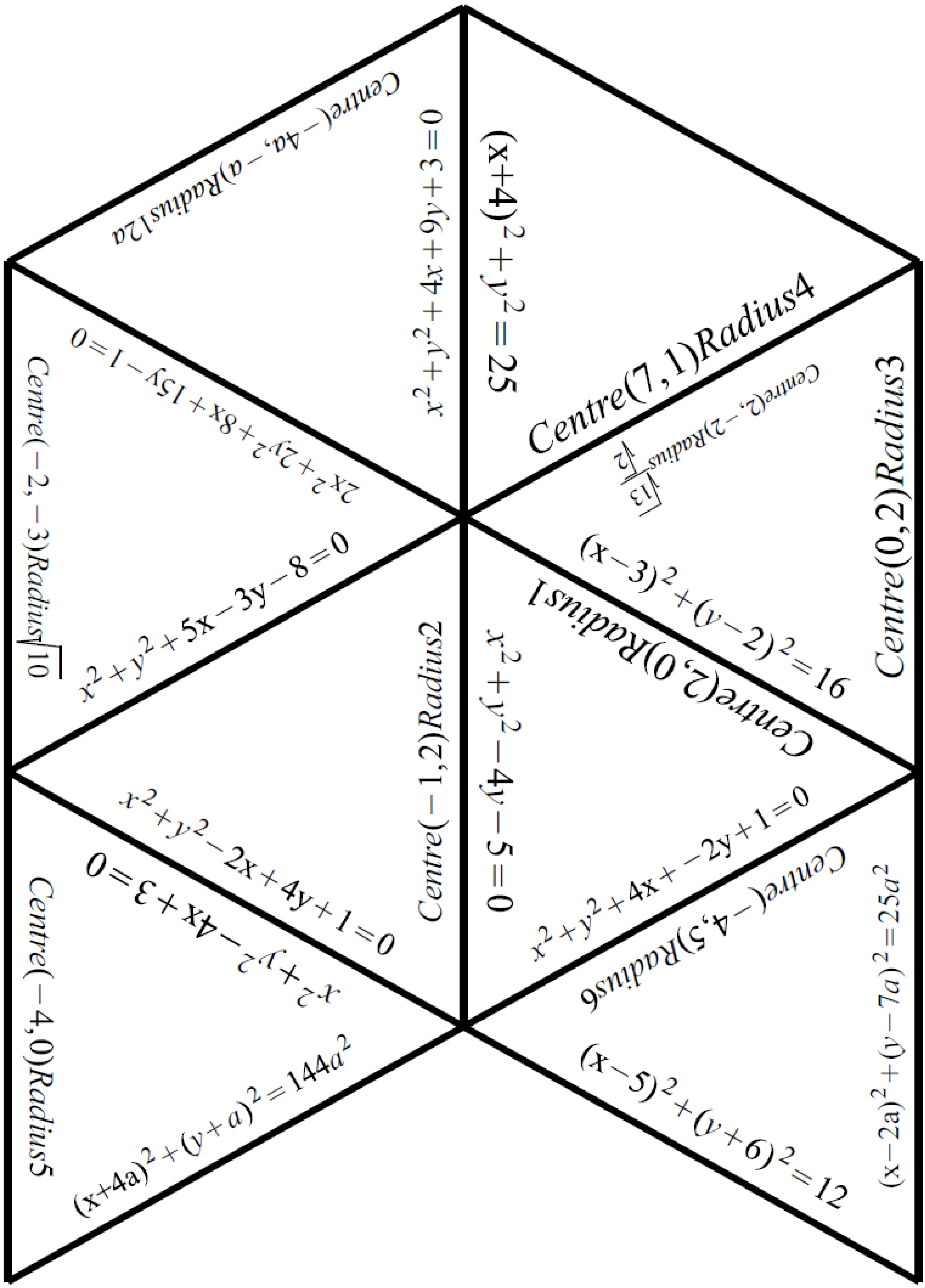
A and B are two vertices of an isosceles triangle. Write down four possibilities of a third vertex.



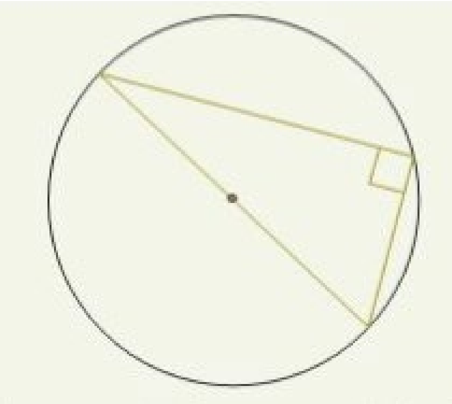
Worksheet H: Tarsia puzzle

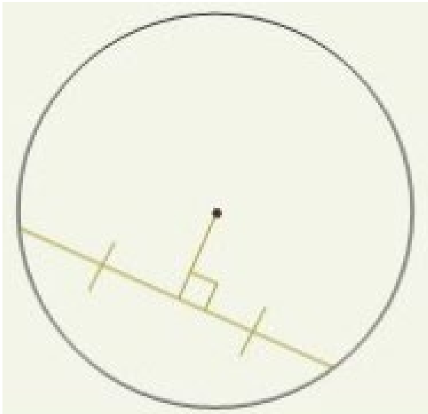


Worksheet H: Tarsia puzzle continued



Worksheet I: Circle theorems





Worksheet J: Goal-free problem

The points P(3, 4) and Q(5, 10) are on the circumference of the circle with centre O(10, 5)

What can you find out from this information?

Worksheet K: Redundant information

While you answer this question, decide whether you actually need all the information.
If not what could be missed out?

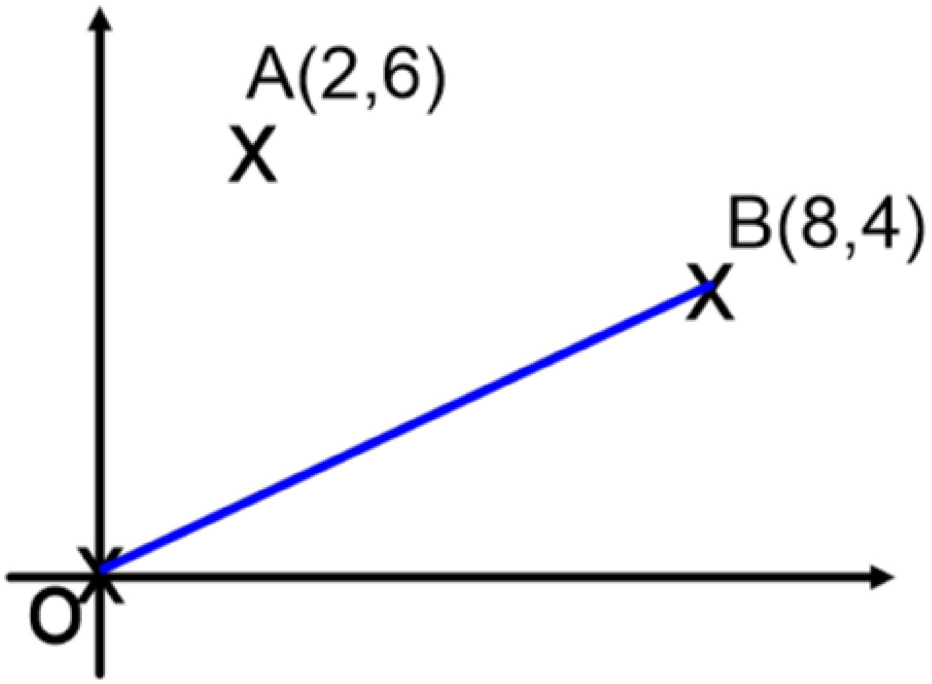
Question:

The points A(6,6), B(6,-2), C(-1,-1) and D(-2,2) are four points on a circle.
Find the equation of the circle.

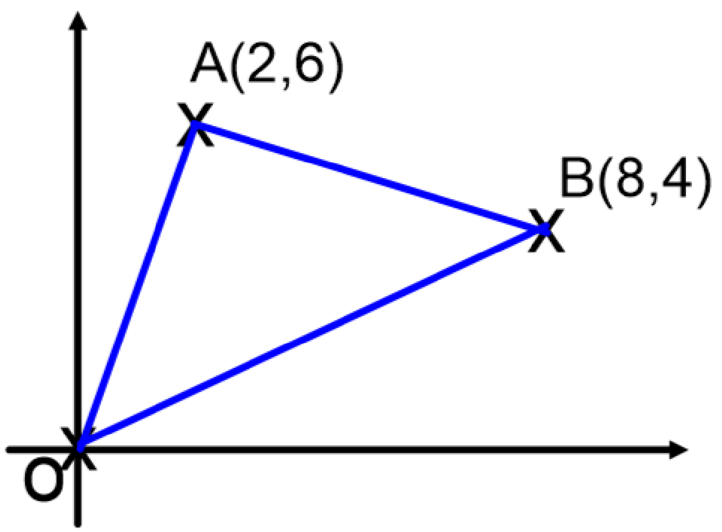
Worksheet L: Building a picture

Show that OB is the diameter of the circle that passes through O(0,0); A(2,6) and B(8,4).

- Draw the information given

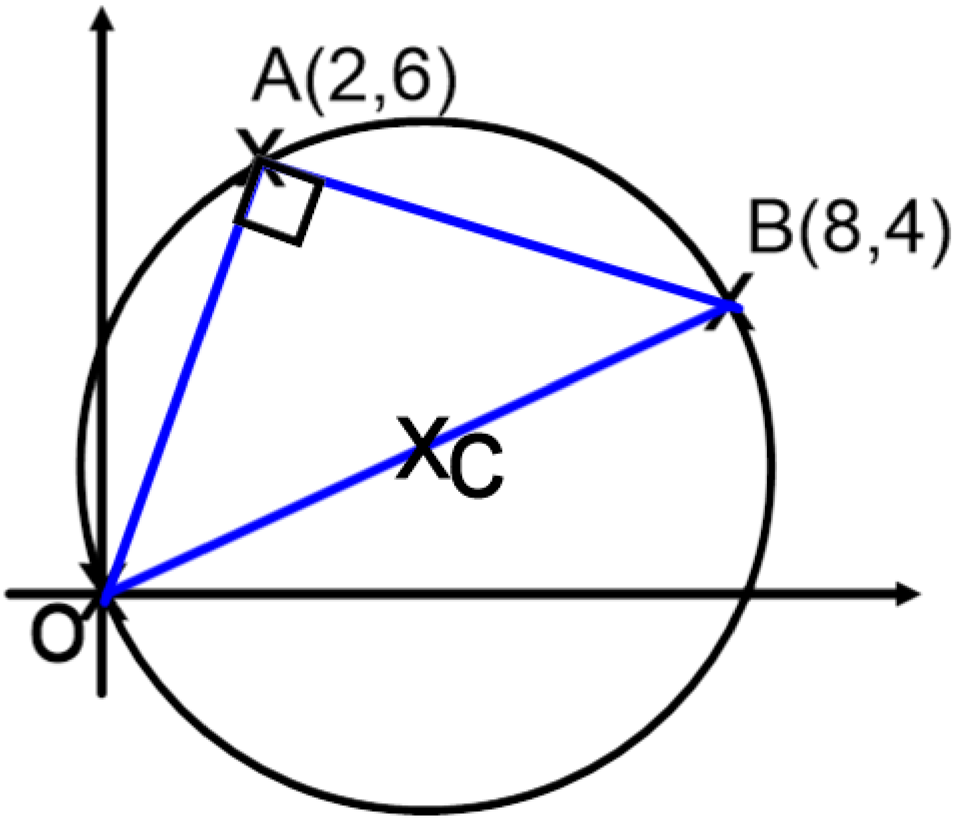


- Think about what would need to be the case for OB to be the diameter



Two chords off the diameter meet at a right angle on the circumference:
∴ OA and AB must be perpendicular
→ the need to find the gradients of each line segment

Worksheet L: Building a picture continued



Gradient OA is 3
Gradient AB is $-\frac{1}{3}$

∴ product of gradients is -1 so perpendicular and angle OAB is 90 degrees,
∴ OAB is the angle in a semi-circle where OB is the diameter.

What else could you find?

The equation of the diameter OB
The equation of the circle.

- Centre,C, is midpoint of OB
- To find the radius there is a choice of OC or CB or half of OB

Worksheet A: Answers

Coordinate Pair	$y=mx+c$	$y - y_1 = m(x - x_1)$	$ax + by + c = 0$
(5,2) and (3,4)	$y = -x + 7$	$y - 2 = -1(x - 5)$	$x + y - 7 = 0$
(9,-1) and (7,2)	$y = \frac{-3}{2}x + \frac{25}{2}$	$y - (-1) = -\frac{3}{2}(x - 9)$	$3x + 2y - 25 = 0$
(-6,1) and (4,0)	$y = \frac{-1}{10}x + \frac{2}{5}$	$y - 1 = \frac{-1}{10}(x - (-6))$	$x + 10y - 4 = 0$
(-12,3) and (5,7)	$y = \frac{4}{17}x + \frac{99}{17}$	$y - 3 = \frac{4}{17}(x - (-12))$	$4x - 17y + 99 = 0$

Worksheet C: Answers

a Parallel **b** Neither **c** Perpendicular

d Perpendicular **e** Neither **f** Parallel

Possible worked solutions

a

$y = 2x + 3$

$y = 2x - 7$

b

$y = 3x$

$2x + y - 3 = 0$
$y - 3 = -2x$
$y = -2x + 3$

c

$y = 4x - 3$

$4y + x = 2$
$4y = -x + 2$
$y = \frac{-x}{4} + \frac{2}{4}$

d

$3x - y + 5 = 0$
$3x = y - 5$
$3x + 5 = y$
$y = 3x + 5$

$x + 3y = 1$
$3y = -x + 1$
$y = \frac{-1}{3}x + \frac{1}{3}$

e

$2x + 5y - 1 = 0$
$5y = -2x + 1$
$y = \frac{-2}{5}x + \frac{1}{5}$

$y = 2x + 7$

f

$2x - y = 6$
$2x - 6 = y$
$y = 2x - 6$

$6x - 3y + 3 = 0$
$-3y + 3 = -6x$
$-3y = -6x - 3$
$y = \frac{-6}{-3}x - \frac{3}{-3}$
$y = 2x + 1$

Worksheet D: Worked solutions

Question 1

Find the equation of the line parallel to $y = 2x + 4$ which passes through (4, 9).

$y = 2x + 4$
The given equation has a gradient $m = 2$
So the line we want also has a gradient of 2 but we do not know c

$$y = 2x + c$$

Substitute the coordinates (4,9) into the equation to find the value of c that makes it work

$$y = 2x + c$$
$$9 = 2 \times 4 + c$$

$$9 = 8 + c$$
$$\therefore c = 1$$

So the required equation is $y = 2x + 1$

Question 2

Find the equation of the line perpendicular to $y = 2x - 3$ which passes through (−2, 5).

$y = 2x - 3$
The given equation has a gradient $m = 2$
So the line we want also has a gradient of

$$-\frac{1}{m} = -\frac{1}{2}$$

but we do not know c $y = -\frac{1}{2}x + c$

Substitute the coordinates (−2, 5) into the equation to find the value of c that makes it work

$$5 = -\frac{1}{2} \times (-2) + c$$

$$5 = 1 + c$$

$$\therefore c = 4$$

So the required equation is $y = -\frac{1}{2}x + 4$

Worksheet D: Worked solutions (continued)

Question 3

A line is perpendicular to the line $2y - x - 8 = 0$, and passes through the point with coordinate (5,-7). Find the equation of the line.

We already have a point on the line
We just need the gradient

The gradient is

$$2y - x - 8 = 0$$
$$2y = x + 8$$
$$y = \frac{1}{2}x + 4$$

Gradient = $\frac{1}{2}$
Gradient of the perpendicular = -2
m = -2
We could use either form of the straight line but using $y - y_1 = m(x - x_1)$ and $(x_1,y_1) = (5,-7)$

$$y - (-7) = -2(x - 5)$$
$$y + 7 = -2x + 10$$
$$\boldsymbol{y = -2x + 3}$$

Worksheet F: Answers

The coordinates of A, B and C are (1,3), (-3,2) and (-2,-2)

No Perpendicular bisector of B and C has a gradient of $-\frac{1}{4}$	**A** AB is perpendicular to BC	**R** The midpoint of A and B is (-1, 2.5)
C The length of AB is $\sqrt{17}$	**S** The gradient of BC is -4	**No** The midpoint of A and C is $(\frac{1}{2}, \frac{1}{2})$
T A and C are points on the line $3y = 5x + 4$	**E** The gradient of AC is $\frac{5}{3}$	**D** Equation of the line through B and C is $y + 4x + 10 = 0$
No AC is perpendicular to AB	**E** Perpendicular bisector of A and C is $5y + 3x - 1 = 0$	**S** The area of triangle ABC is 8.5cm^2

René Descartes

He made an important connection between geometry and algebra, which allowed for the solving of geometrical problems by algebraic equations.

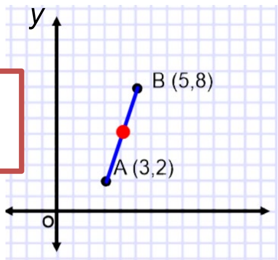
Worksheet G: Answers

1

Work out the coordinates of the midpoint of AB.

$\left(\frac{3+5}{2}, \frac{2+8}{2}\right)$

$(4,5)$



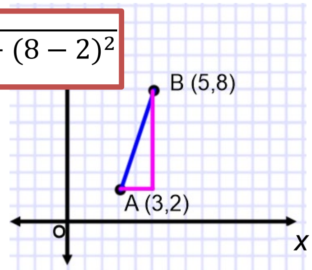
2

Find the length of the line AB.

$\sqrt{(5-3)^2+(8-2)^2}$

$\sqrt{40}$

$2\sqrt{10}$



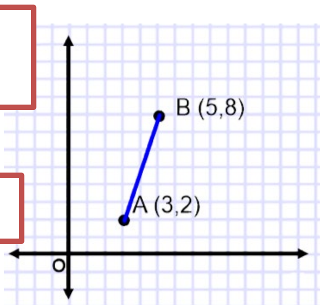
3

$m = 3$

Work out the gradient of the line of AB.

$m = \frac{8-2}{5-3}$

$m = 3$



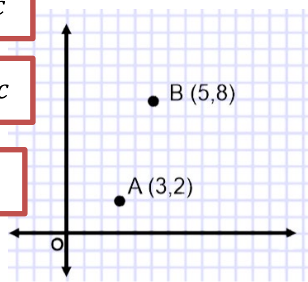
4

Work out the equation of the line passing through A and B.

$y = mx + c$

$2 = 3(3) + c$

$y = 3x - 7$



5

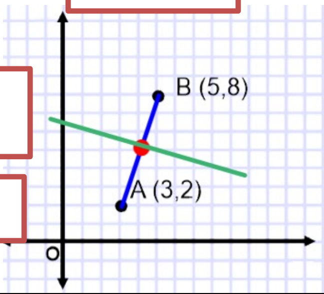
Work out the equation of the perpendicular bisector of AB.

Midpoint $(4,5)$

$m = -\frac{1}{3}$

$y - 5 = -\frac{1}{3}(x-4)$

$3y + x - 19 = 0$

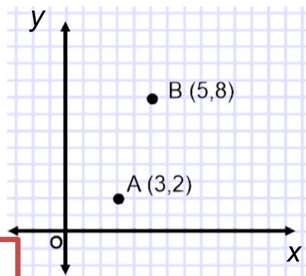


6

A and B are two vertices of an isosceles triangle. Write down four possibilities of a third vertex.

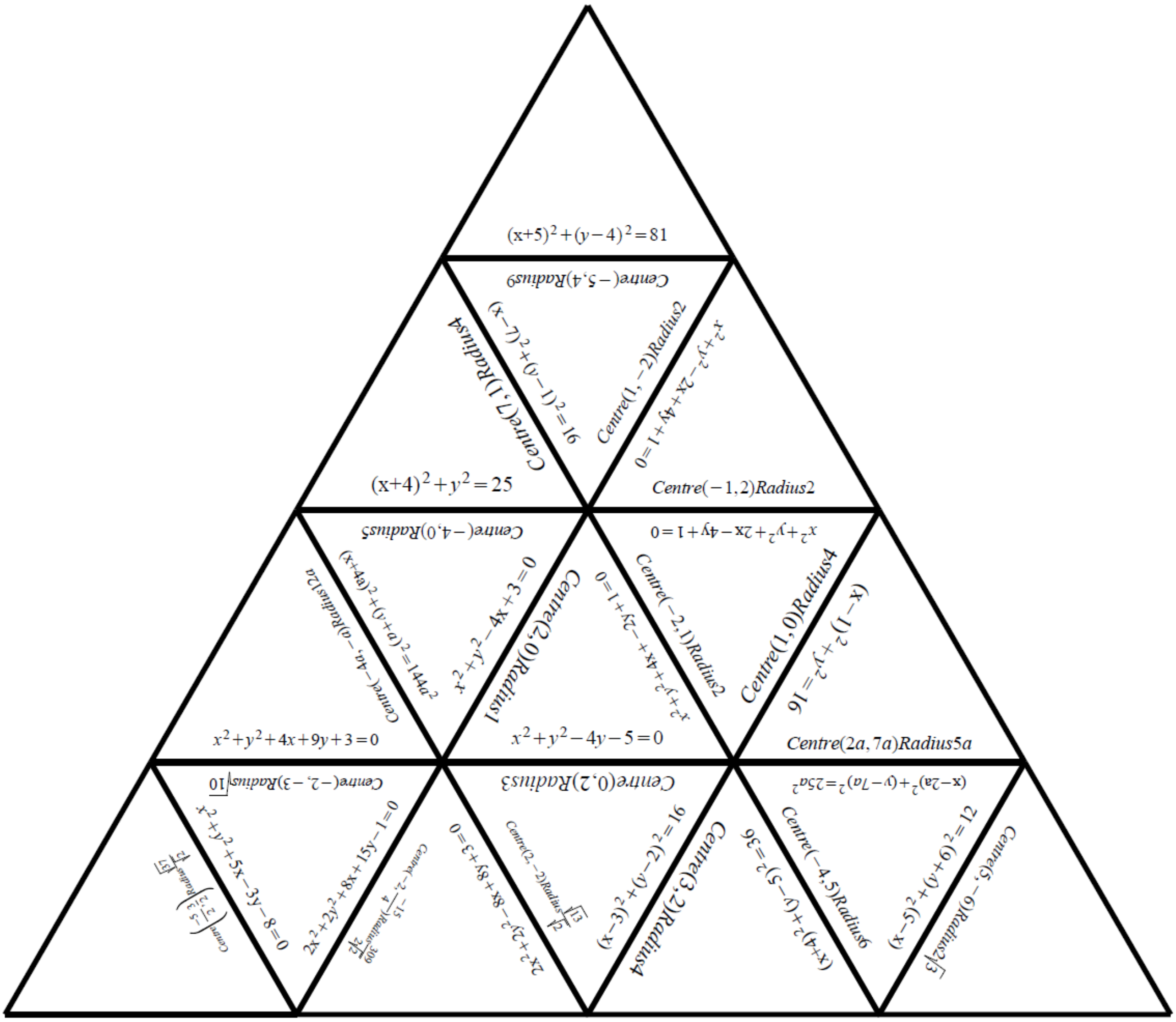
Any point that lies on the line with equation

$3y + x - 19 = 0$

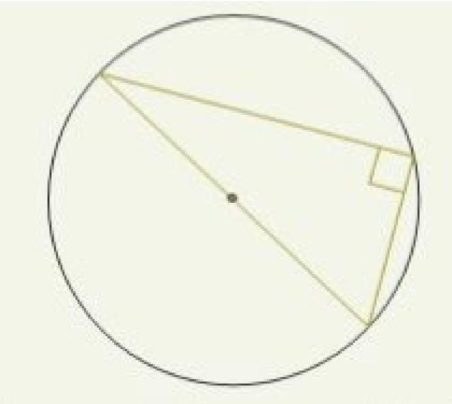


or (7,2)

Worksheet H: Answers

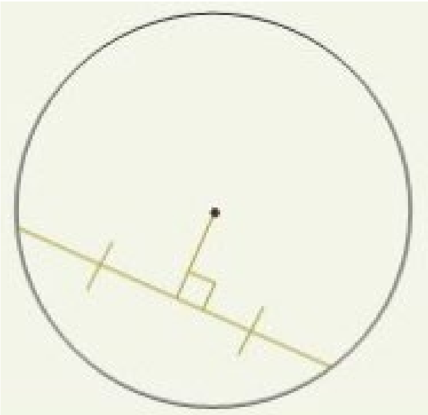


Worksheet I: Answers



The angle subtended from the diameter meets the circumference at 90 degrees

The perpendicular bisector of a chord passes through the centre of the circle



A radius and tangent to the same point meet at 90 degrees

Worksheet J: Answers

Some of the information that learners will find is:

the radius? $\sqrt{50}$

the midpoint of the chord? M(4,7)

the length of the chord? $\sqrt{40}$

the equation of the perpendicular bisector? $x + 3y - 25 = 0$

the equation of the circle? $(x - 10)^2 + (y - 5)^2 = 50$

PQ is the same length as OM? $\sqrt{40}$

the area of the triangle POQ?
20 units squared $\frac{\sqrt{40}\sqrt{40}}{2}$ because of the right angle

Worksheet K: Answers

To find the equation of a circle we need the centre and the radius.

To find the centre:

Use the fact that the perpendicular bisector of a chord passes through the centre.

We have 4 points so there are many chords to choose from.

If we choose AB the equation of the perpendicular bisector is $y = 2$

If we choose CD the equation of the perpendicular bisector is $y = \frac{1}{3}x + 1$

These two equations meet at (3,2). Therefore the centre is (3,2)

To find the radius:

Use the distance from the centre to any one of the points on the circle.

The radius is 5

The equation of the circle is therefore $(x-3)^2 + (y-2)^2 = 25$

Redundant information:

It was necessary to have two chords but using AB and BC or BC and CD would have given the required information so one of the points was not needed.

Three points uniquely define a circle.

Two points could have a number of circles drawn through them.
Try (0,0) and (0,6) as an example.

Cambridge Assessment International Education
The Triangle Building, Shaftesbury Road, Cambridge, CB2 8EA, United Kingdom
t: +44 1223 553554
e: info@cambridgeinternational.org   www.cambridgeinternational.org