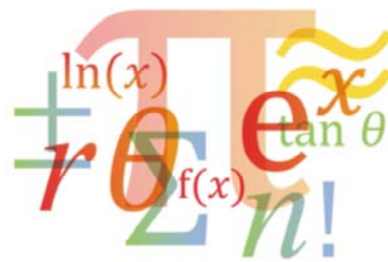
Cambridge Assessment
International Education

How to Guide

How to … write proofs



Version 1

Cambridge Pathway

Why is writing proofs important for learners?

Proof is an essential feature of mathematics, especially pure mathematics. Writing proofs

- helps learners to organise their ideas
- helps learners to deepen their understanding of mathematical relationships
- helps learners to develop their ability to argue mathematically
- encourages learners to communicate clearly with someone else in writing

The new 9709 syllabus has mathematical communication (including Proof) as one of its three Key Concepts. This means that proof plays a vital role in topics throughout the syllabus and learners need to become proficient at producing complete proofs using accurate mathematical notation.

For more information on mathematical writing in general, please see ‘How to … write mathematically’. This guide concentrates on formal proof.

How do you know it's a proof?

- Some common phrases are used in mathematics questions. These indicate that learners need to construct a logical argument, showing all the steps, leading to a particular statement:

  'Prove that …'

  'Show that …'

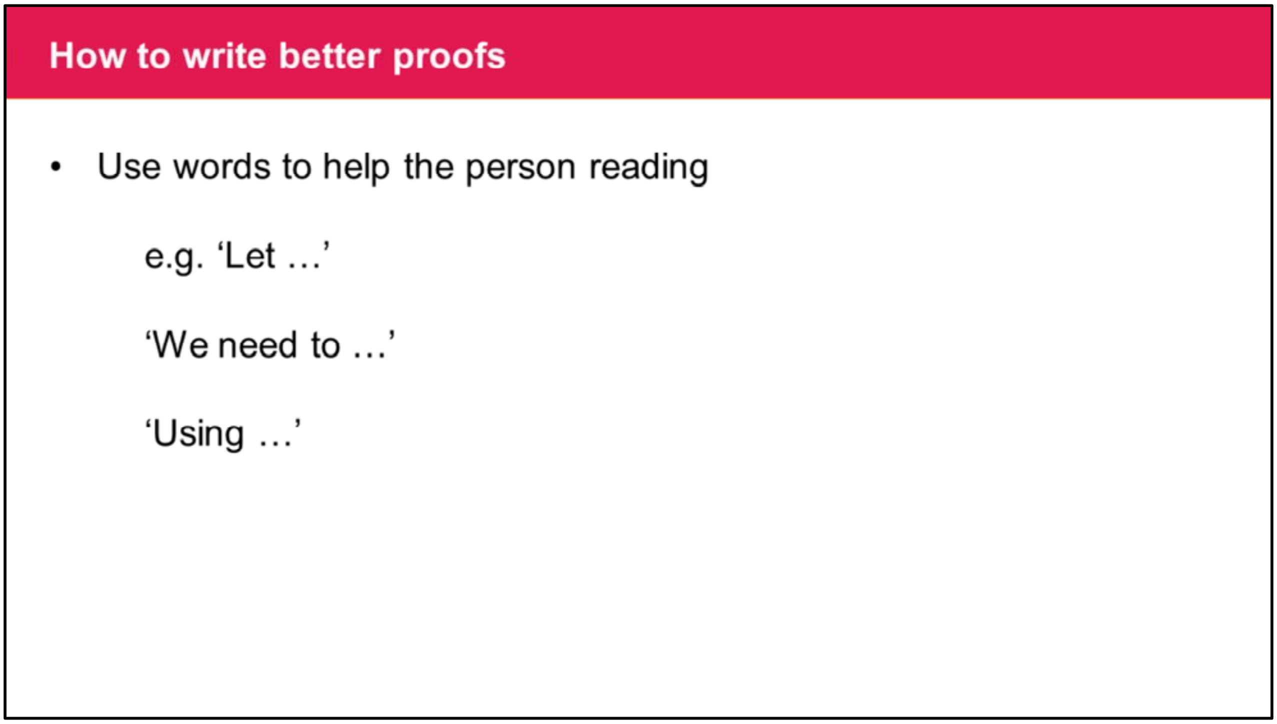
'Verify' does not require a proof. It usually means that you substitute in a certain value to obtain the given result, e.g. 'Verify that there is a stationary point when $x = 2$' means substitute $x = 2$ into the gradient function, $\mathrm{d}y/\mathrm{d}x$, obtain 0 then draw a conclusion.

How to organise a proof

- Start with the left hand side and aim towards the right hand side (target)
- Use equals signs appropriately e.g. at the beginning of each line
- Write a sequence of logical steps where each step follows logically from the previous step, with no gaps
- Use sufficient detail to justify each step
- The chain of reasoning must lead to the conclusion written exactly as given in the question

e.g. Show that $\cot x + \tan x = \frac{1}{\sin x \cos x}$.

The proof starts with $\cot x + \tan x =$ … and the sequence of steps should end with $\frac{1}{\sin x \cos x}$, the exact statement given in the question. Learners can view the right hand side as a target to aim for. It will often give them clues about how to reach it. In this example, they can spot that it contains sin*x* and cos*x* so it would be sensible to begin the proof by writing cot*x* and tan*x* in terms of sin*x* and cos*x*.



Examples of statements that could help the reader to follow a proof:

- Let $a = x^2$
- We need to show that the lines are perpendicular
- Using $\cos^2 x + \sin^2 x = 1$
- Using the Factor Theorem, …

Common mistakes when doing proofs

- Assuming that the statement to be proven is true (going round in circles)
- Starting with both sides and ending up with $1 = 1$ or $0 = 0$
- Missing out vital steps in the working so there is a gap in the logic
- Not concluding with the statement they are trying to prove
- Using notation incorrectly
  e.g. $\Rightarrow$ or $\rightarrow$ at the start of each line when they should use $=$

Learners may be used to writing down an algebraic expression from the question e.g. when solving equations. They should not do the same when writing a proof because the expression in the question needs to be their concluding statement.
Rough working is a good idea, and working backwards may help learners to prepare a proof. However the ‘real’ proof needs to be written out forwards and lead to the correct conclusion. All the rough working and working backwards should be crossed out.
Learners should check their proof carefully to make sure that no steps are left out.
They should get into the habit of concluding with the statement they are trying to prove – it can be satisfying to know they have arrived!

Important points

- Learners may quote formulae from the formula book unless they are asked to prove them
- If they reach a dead end and start again, learners should cross out the earlier attempt(s)

For example, $sec^2 x$ as the derivative of tan$x$ can be used unless the question specifically asks for a proof that it is the derivative.

How you can help learners to get better at writing proofs 1

- Demonstrate to them a variety of standard proofs during their lessons even if they will not be required to replicate them.
  - Seeing what lies behind a mathematical fact or formula will deepen their understanding of it
  - They will learn about the discipline of proof, a topic that may be new to them prior to the AS & A Level course
- Ask them to analyse a proof from a textbook or other source
- Ask them to evaluate each other's proofs (peer review)

Examples of proofs that can be incorporated into the AS & A Level course:

- proof by contradiction that $\sqrt{2}$ is irrational,
- proof of trigonometric addition formulae,
- proof of the product and quotient rules.

There are many other situations where proving a formula helps learners to learn the formula and apply it more confidently in different problem-solving situations.

How you can help learners to get better at writing proofs 2

Proof sorting:

- Give learners printed proofs cut up into individual cards (one card for each step) and ask them to sort the cards into the correct order
- A variation on the proof sort is to miss out one vital step. Ask learners to write on a blank card the statement that they would add
- After they have completed a proof sort, you could ask learners to put away the cards then write out the proof in their notes
- Or you could give them a printed version of the proof to annotate the steps, explaining what is happening in each line

Proof sorting activities are ideally suited to learners working in pairs or groups. There are some good examples on Underground Mathematics: go to https://undergroundmathematics.org/ and search for Proof sort. You will be able to print out the cards for proofs such as:

- Irrationality of $\sqrt{2}$
- Quadratic formula
- Laws of logarithms
- Trigonometric formulae

It is also quite easy to create your own cards for proofs in other topics. These activities are useful for helping students to think logically as they need to study the steps carefully and understand what is being done mathematically.

Further reading for learners progressing to mathematical degrees

These books will help learners to see the rigour required for higher level study of the subject. Proof is a high priority in both of the books which will prepare them for more formal written work.

- Lara Alcock 'How to study for a mathematics degree' (2013) published by Oxford University Press
- Kevin Houston 'How to think like a mathematician' (2009) published by Cambridge University Press

Learners who are confident with writing proofs at AS & A Level will find it easier to progress to a degree with a high mathematical content.