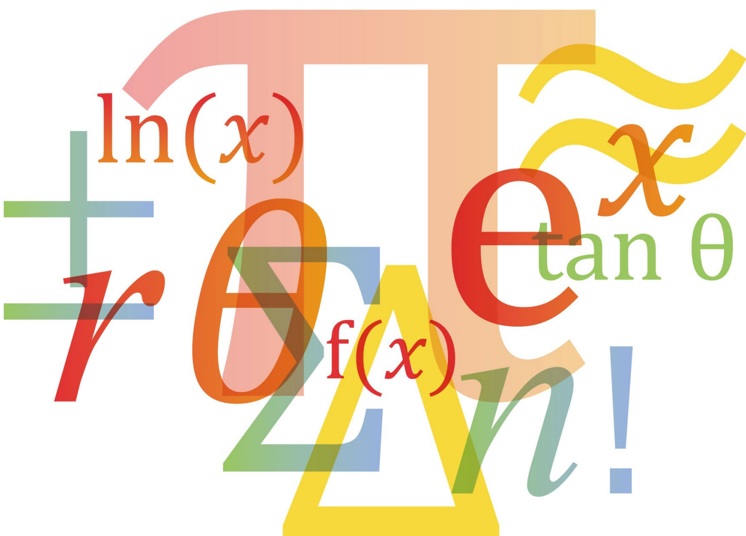
Cambridge Assessment International Education

Teacher Pack

Complex Numbers

Cambridge International AS & A Level

Mathematics 9709



Version 0.1

Cambridge Pathway

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Introduction

This pack will help you to develop your learners' skills in mathematical thinking and mathematical communication, which are essential for success at AS & A Level and in further education.

Mathematical thinking and communication will be developed by focussing on:

1. Conceptual understanding – the 'why' behind the 'what'
2. Strategic competence – forming and solving problems
3. Adaptive reasoning – explanations, justifications and deductive reasoning

Throughout all activities, the learners will also develop:

- Procedural fluency – know when, how and which rules to use
- Positive disposition – believe maths can be learned, applied and is useful
- Their skills in writing mathematically – writing working & proofs

These link to the course Assessment Objectives (AOs) which you can find in detail in the syllabus:

A01 Knowledge and understanding

A02 Application and communication

Each *Teacher Pack* contains one or more lesson plans and associated resources, complete with a section of preparation and reflection.

Each lesson is designed to be an hour long but you should adjust the timings to suit the lesson length available to you and the needs of your learners.

Important note
Our *Teacher Packs* have been written by **classroom teachers** to help you deliver topics and skills that can be challenging. Use these materials to supplement your teaching and engage your learners. You can also use them to help you create lesson plans for other topics.

This content is designed to give you and your learners the chance to explore a more active way of engaging with mathematics that encourages independent thinking and a deeper conceptual understanding. It is not intended as specific practice for the examination papers.

The *Teacher Packs* are designed to provide you with some example lessons of how you might deliver content. You should adapt them as appropriate for your learners and your centre. A single pack will only contain at most four lessons, it will **not** cover a whole topic. You should use the lesson plans and advice provided in this pack to help you plan the remaining lessons of the topic yourself.

Lesson preparation

This *Teacher Pack* will cover the following syllabus content.

Candidate should be able to:	Notes and examples
• understand the idea of a complex number, recall the meaning of the terms real part, imaginary part, modulus, argument, conjugate, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal	Notations $\mathrm{Re}\, z$, $\mathrm{Im}\, z$, $\lvert z \rvert$, $\arg z$, z^* should be known. The argument of a complex number will usually refer to an angle θ such that $-\pi < \theta \leq \pi$, but in some cases the interval $0 \leq \theta < 2\pi$ may be more convenient. Answers may use either interval unless the question specifies otherwise.
• carry out operations of addition, subtraction, multiplication and division of two complex numbers expressed in Cartesian form $x + \mathrm{i}y$	For calculations involving multiplication or division, full details of the working should be shown.
• use the result that, for a polynomial equation with real coefficients, any non-real roots occur in conjugate pairs	e.g. in solving a cubic or quartic equation where one complex root is given.
• represent complex numbers geometrically by means of an Argand diagram	
• carry out operations of multiplication and division of two complex numbers expressed in polar form $re^{i\theta} = r(cos\,\theta + \mathrm{i}\, sin\,\theta)$	Including the results $\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \lvert z_2 \rvert$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$, and corresponding results for division.
• find the two square roots of a complex number	e.g. the square roots of $5 + 12\mathrm{i}$ in exact Cartesian form. Full details of the working should be shown.
• understand in simple terms the geometrical effects of conjugating a complex number and of adding, subtracting, multiplying and dividing two complex numbers	
• illustrate simple equations and inequalities involving complex numbers by means of loci in an Argand diagram	e.g. $\lvert z - a \rvert < k$, $\lvert z - a \rvert = \lvert z - b \rvert$, $\arg(z - a) = \alpha$.

Prior knowledge and skills

For all lessons, it is assumed that learners have already completed Cambridge IGCSE™ Mathematics 0580, or a course at an equivalent level. See the syllabus for more details of the expected prior knowledge for taking Cambridge International AS & A Level Mathematics 9709.

When planning any lesson, make a habit of always asking yourself the following questions about your learners' prior knowledge and skills:

- Do I need to re-teach this or do learners just need some practice?
- Is there an interesting activity that will efficiently achieve this?

Key learning objectives

The following list represents the main underlying concepts that you should make sure your learners have understood by the end of this topic.

- understand the idea of a complex number, recall the meaning of the terms real part, imaginary part, conjugate, modulus and argument
- use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal
- carry out operations of addition, subtraction, multiplication and division of two complex numbers expressed in Cartesian form $x + \mathrm{i}y$
- use the result that, for a polynomial equation with real coefficients, any non-real roots occur in conjugate pairs
- find the two square roots of a complex number
- represent complex numbers geometrically by means of an Argand diagram
- carry out operations of multiplication and division of two complex numbers expressed in the polar form $r(cos\,\theta + \mathrm{i}\,sin\,\theta) = re^{\mathrm{i}\theta}$
- illustrate simple equations and inequalities involving complex numbers by means of loci in an Argand diagram

Why this topic matters

Complex numbers are an essential part of mathematics because they extend the concept of real numbers and provide a deeper understanding of algebra, geometry, and calculus.
1. Solving Polynomial Equations
2. Enhancing Understanding of Algebra
3. Applications in Geometry & Trigonometry
4. Use in Calculus & Differential Equations in Euler's formula and Fourier Transformation in Signal Processing
5. Preparation for Higher Mathematics & STEM Fields: for example, in complex analysis, AC circuit and fluid dynamics

Key terminology and notation

Your learners will need to be confident with the following terminology and notation.

Real Part: a is called the real part of the complex number $a + b\mathrm{i}$
Imaginary Part: b is called the imaginary part of the complex number $a + b\mathrm{i}$
Conjugate: $a - b\mathrm{i}$ is called the conjugate of $z = a + b\mathrm{i}$, denoted as $z^* = a - b\mathrm{i}$
Modulus: magnitude of the vector (r or $|z|$) in Argand Diagram
Argument: angle of vector formed with positive x axis (θ or $\arg z$) in Argand Diagram
Cartesian Form: $z = a + b\mathrm{i}$
Polar Form: $z = r(\cos\theta + \mathrm{i}\sin\theta)$
Locus: the region where the set of points with specific properties locate

Lesson progression

Lesson 1 covers the basic terminology and definition of complex number and its algebra, including addition, subtraction, multiplication and division. Lesson 2 then move onto the polynomial function and its roots with conjugate pair theorem. Lesson 3 is about expressing complex number geometrically in Argand diagram with modulus and argument. Polar form of complex number is introduced and product and division in polar form is explained. Lesson 4 focuses on the locus of complex number in Argand diagram, which is an very important type question in the A Level mathematics examination.

Going forward

This topic is important in further university study in mathematics or physics.

Lesson 1: Introduction to Complex Number

Preparation	
Resources	• White Board • Lesson 1 slides • Worksheet 1a and 1b
Learning objectives	By the end of this lesson, all learners should be able to: • understand the idea of a complex number, recall the meaning of the terms real part, imaginary part, and conjugate; • use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal • carry out operations of addition, subtraction, multiplication of two complex numbers expressed in Cartesian form $x + \mathrm{i}y$. By the end of this lesson, most learners should be able to: • understand the idea of a complex number, recall the meaning of the terms real part, imaginary part, and conjugate; • use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal • carry out operations of addition, subtraction, multiplication and division of two complex numbers expressed in Cartesian form $x + \mathrm{i}y$

Dependencies

For all lesson plans in this *Teacher Pack*, knowledge from the following 9709 syllabus content is useful and/or required.

Candidate should be able to:	Notes and examples
• solve quadratic equations in one unknown	By factorising, completing the square and using the formula.

Timings	Activity
5 min	**Starter/Introduction** Teach this lesson use the Lesson 1 slides. As a starter, display **side 2** and ask the learner this question, Solve $x^2 + 1 = 0$
45 min	**Main lesson** Part I: Display **slide 3** and stress on the definition of $\mathrm{i}^2 = -1$. Challenge: ask them is there any difference between $\mathrm{i}^2 = -1$ and $\mathrm{i} = \sqrt{-1}$ Real part and imaginary part are important concepts and make sure all learners understand that both of them refer to the real number a and b.

Timings	Activity
	Go over **slides 4~5** of the addition, subtraction and multiplication of complex number. Emphasise on all the bracket expansion formula can be used in complex number. Guide them on how to expand the bracket during teaching of products. Example on **slides 6 and 7** should be accessible and straightforward to all learners. Display **slide 8 and 9** with the conjugate and the division. Depend on the student level, division could be challenging for the weaker learners. Link this to rationalisation of the denominator in surds (pure math 1) if possible, which might remediate the problem. With teacher's guidance, the example on **slide 10 and 11** should be accessible to most learners. Give the learner **worksheet 1a** for 15~20 minutes. It is worth to go over the last question in the worksheet 1a. Part II: Though trivial, it is worth to mention the identical complex number in **slide 12** and its meaning and application in solving the simultaneous equation of complex numbers. Most of the learners will find it tedious until they have seen how to apply it on **slide 13**. Support: This example must be explained in detail. Make sure that you guide the learners on this example because they will soon get lost on the complicated process. Give the learner **worksheet 1b** for 10~15 minutes. The two questions are very challenging so leave the unfinished questions as homework.
5 min	**Plenary** Wrap up: **Slide 14** True or false questions and refresh their minds from doing complex number questions. Some of them could be confusing.

Reflection Reflect on your lesson, use the **Lesson reflection** notes to help you.

Lesson plan 2: Polynomial Function with Complex Root

Preparation		
Resources		• White Board • Lesson 2 slides • Worksheet 2a, 2b
Learning objectives	By the end of the lesson:	• use the result that, for a polynomial equation with real coefficients, any non-real roots occur in conjugate pairs • find the two square roots of a complex number

Dependencies

For all lesson plans in this *Teacher Pack*, knowledge from the following 9709 syllabus content is useful and/or required.

Candidate should be able to:	Notes and examples
• solve quadratic equations in one unknown	By factorising, completing the square and using the formula.

Timings	Activity
5 min	**Starter/Introduction** If you need to go over the **worksheet 1b**, consider to take 5~10 minutes out from this class and address them. Show **slide 2 and 3**, review the quadratic formula and the discriminant with learners, then link the lesson 1 material to here. If learners struggled with the quadratic formula (which they should not), consider go over one example with them.
45 min	**Main lesson** Part I: Show learners the example on **slide 3**, ask them to apply the quadratic formula first and see whether they could figure out the answer by themselves. This question should be straightforward. After obtaining the 2 solutions, ask them the relationship between the answer. Stronger learners should be able to answer that 2 solutions are conjugate pairs of each other. Distribute the **worksheet 2a** and solve question 1. Ask them what they find the relationship between the 2 solutions again. Let them guess the conjugate pair theorem. Transition smoothly to conjugate pair theorem. Show them the example on **slides 6~8**. Review long division if necessary.

Timings	Activity
	Challenge: The stronger learners should be able to understand the approach 2 on slide 8. (approach 2 on slide 8 is optional). Ask the leaners to finish question 2 on the **worksheet 2a**. Part II: The square root of complex number should be accessible for all learners. Link this to the lesson 1: multiplication of the complex number. Show them the example on slide 10~11. The solving of x and y can be challenging for some of the learners. If they have to use the substitution, allow them. Direct observation is straightforward and ask them to try integers solutions first. Remind them there must be 2 roots! Exactly the same with $x^2 = 9$. Distribute **worksheet 2b** and ask them to finish in class. Challenge: the question 2 on **worksheet 2b** is optional only for the stronger learners.
5 min	**Plenary** Wrap up: **Slide 12** Link to the Argand diagram which is to be taught in Lesson 3.

Reflection	Reflect on your lesson, use the **Lesson reflection** notes to help you.

Lesson plan 3: Argand Diagram and Other Forms of Complex Numbers

Preparation	
Resources	• White Board • Lesson 3 slides • Worksheet 3a, 3b
Learning objectives	By the end of this lesson, all learners should be able to: • understand the idea of modulus and argument • represent complex numbers geometrically by means of an Argand diagram By the end of this lesson, most learners should be able to: • understand the idea of modulus and argument • represent complex numbers geometrically by means of an Argand diagram • carry out operations of multiplication and division of two complex numbers expressed in the polar form $r(cos\,\theta + \mathrm{i}\,sin\,\theta) = e^{\mathrm{i}\theta}$ By the end of this lesson, some learners should be able to: • understand the idea of modulus and argument • represent complex numbers geometrically by means of an Argand diagram • carry out operations of multiplication and division of two complex numbers expressed in the polar form $r(cos\,\theta + \mathrm{i}\,sin\,\theta) = re^{\mathrm{i}\theta}$ • understand in simple terms the geometrical effects of multiplying and dividing two complex numbers

Dependencies

For all lesson plans in this *Teacher Pack*, knowledge from the following 9709 syllabus content is useful and/or required.

Candidate should be able to:	Notes and examples
• carry out addition and subtraction of vectors, and interpret these operations in geometrical terms • calculate the magnitude of a vector	

Timings	Activity
5 min	**Starter/Introduction** Show them **slide 1** to remind the learners about definition of angles in coordinates and unit circles. This lesson requires them to be familiar with the trigonometric functions and special angle values. Review with them if they need.
65 min	**Main lesson** Part I: Show them **slides 2~4**. Link with the probing question at the end of Lesson 2, expressing the complex number on the complex plane (just a normal xy plane). More

Timings	Activity
	examples could be given to the students to remind them how to draw vectors in different coordinates. The definition of modulus and argument is key, using easy words to define those complicated terms. For example, modulus is the length and argument is the angle, which will help the learners to be less "afraid of" the math terminologies. Challenging: Remind them that the calculation of the argument (**slide 4**) has different modification by π is because of the domain of argument is from $-\pi < \arg z \leq \pi$. It is very important for them to draw the complex number if the questions ask them to find the argument because $\tan^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Show learners about the example on **slide 6** and go over the 2 questions in details. Then, ask them to complete the question 1 on the **worksheet 3a**. Remind them again about the importance of drawing the complex number on the Argand diagram before they calculate the argument. Part II: Show **slide 8** and illustrate the polar form of the complex number by drawing the diagram, especially when illustrating $x = r\cos\theta$ and $y = r\sin\theta$. Show **slide 9~10** and go over the questions with the special angles. Then, ask them to complete the question 2 on the **worksheet 3a**. **Slide 11 and 12** are the same rule in the different way. Choose one to show the learners – whichever one works for you. These rules could be challenging to understand. It is better to review these rules after learning of the $re^{\mathrm{i}\theta}$ form. (**slide 17** is put in the teacher resource for this purpose) **Slide 13~15** is challenging for most of the leaners – understanding the product and division of complex number geometrically. Remind them the meaning of modulus and argument – modulus is the length and argument is the angle. Therefore, increase in modulus means length enlargement and increase of angle means rotation anticlockwise (vice versa). Use the Argand diagram on **slide 15** to illustrate. Support: If all learners still seem to struggle, skip this part and skip Question 1 on the **worksheet 3b.** Part III: Show **slide 16 and 18**. Learners does not have to know the de Moivre's theorem, but they need to know $re^{\mathrm{i}\theta} = r(\cos\theta + \mathrm{i}\sin\theta)$. **Slide 18** proof can be used for leaners to explore by themselves but in this lesson, we hardly have any time for doing this. Show **slide 19~20**. If learners can figure out the correct modulus and argument, then they would not have problem of calculating the power. Ask them to finish question 2 on **worksheet 3b** as homework if no time left.
5 min	**Plenary**

Timings	Activity
	There is hardly any time left for the wrap up. Review the definition of modulus and argument and the product and division of complex numbers.

Reflection	Reflect on your lesson, use the **Lesson reflection** notes to help you.

Lesson plan 4: Locus in Argand Diagram

Preparation	

Resources	• White Board • Lesson 4 slides • Worksheet 4

Learning objectives	By the end of this lesson, most learners should be able to: • illustrate simple equations and inequalities involving complex numbers by means of loci in an Argand diagram

Dependencies

Learners need to know representing angles and presenting vector from IGCSE mathematics. Learners are required to understand the material in lesson 1, and 3.

Timings	Activity
15 min	**Starter/Introduction** Teach this lesson use the Lesson 4 slides. Show **slide 2**, and review the Argand Diagram, modulus, and argument with learners.
45 min	**Main lesson** Part I: **Slide 3 and 4** demonstrates the basic form of locus, including the real part, imaginary part and the argument. Make sure all learners are able to represent these in the Argand diagram. **Slide 5** link the geometric meaning of subtraction in complex number with the vectors. Therefore, **slide 6** demonstrates the type II of locus in the form of $\arg(z - w) = \theta$. This starts to be challenging and for the weak students they might have to memorise this conclusion. Part II: **Slide 7~10** is going to be illustrating the locus in the form of $\lvert z - w\rvert = \lvert z - p\rvert$ and $\lvert z - w\rvert \leq r$. It is very important for the learners to understand the reason why the locus is perpendicular bisector between WP and a circle correspondingly. Therefore, a full detailed explanation is provided in the slides. Having said that, it is difficult for the learners without strong vector background to visualise it. Therefore, an algebraic approach is also provided in **slide 9 and 10**. The proof on those two slides can be used if the learners is weak in geometry but strong in algebra. Part III:

Timings	Activity
	Slides 11~17 are all about exam style questions. This locus type of question is very common in the Cambridge A level math exam. Therefore, it is very important for teachers to prepare students in familiarisation of exam style questions. You could alter the question if you think there are more proper practice questions for your students. After going over at least 2 examples, distribute the **worksheet 4** and ask the learners to finish them in class. This worksheet is very challenging. Most students should be able to access the first question.
5 min	**Plenary** Address any question arises during worksheet 4 Present **slide 18** and ask the learners to note down all the different possible type of locus question in the exam.

Reflection	Reflect on your lesson, use the **Lesson reflection** notes to help you.
	How the learners react with the difficulty of question in worksheet 4? How the learners react with maximum argument part in the worksheet 4?

Planning your own lessons

You now need to plan lessons to cover the following bullet points:

Candidate should be able to:	Notes and examples
• use algebraic methods to solve problems involving lines and circles	Including use of elementary geometrical properties of circles, e.g. tangent perpendicular to radius, angle in a semicircle, symmetry. Implicit differentiation is not included.
• understand the relationship between a graph and its associated algebraic equation, and use the relationship between points of intersection of graphs and solutions of equations.	e.g. to determine the set of values of k for which the line $y = x + k$ intersects, touches or does not meet a quadratic curve.

Follow the structure of the *Skills Pack*, and use techniques from the 'How to' guides, to create your own engaging lessons to cover these bullet points. Consider what preparation you need for each lesson: what prior knowledge is needed, what are the key objectives, what are the dependencies, what common misconceptions are there, and so on.

Below, we have provided an outline of some activities and approaches you might like to try.

Lesson *x*: Solving problems with lines and circles
Common misconceptions:
Starter: You could try …
Main: You could use resource xxxxx and have a class discussion about *y* ….
Plenary: You could try …

Lesson *y*: The relationship between a graph and its algebraic equation
Common misconceptions:
Starter: You could try …
Main: You could use resource xxxxx and have a class discussion about *y* ….
Plenary: You could try …

You will find some other activity suggestions in the Scheme of Work.

Lesson reflection

As soon as possible after the lesson you need to think about how well it went.

One of the key questions you should always ask yourself is:

Did all learners get to the point where they can access the next lesson? If not, what will I do?

Reflection is important so that you can plan your next lesson appropriately. If any misconceptions arose or any underlying concepts were missed, you might want to use this information to inform any adjustments you should make to the next lesson.

It is also helpful to reflect on your lesson for the next time you teach the same topic. If the timing was wrong or the activities did not fully occupy the learners this time, you might want to change some parts of the lesson next time. There is no need to re-plan a successful lesson every year, but it is always good to learn from experience and to incorporate improvements next time.

To help you reflect on your lesson, answer the most relevant questions below.

Were the lesson objectives realistic?
What did the learners learn today? Or did they learn what was intended? Why not?
What proportion of the time did we spend on the most important topics?
Were there any common misconceptions?
What do I need to address next lesson?
What was the learning atmosphere like?
Did my planned differentiation work well?
How could I have helped the lowest achieving learners to do more?
How could I have stretched the highest achieving learners even more?
Did I stick to timings?
What changes did I make from my plan and why?

Summary evaluation

What two things went really well? (Consider both teaching and learning.)

What two things would have improved the lesson? (Consider both teaching and learning.)

What have I learned from this lesson about the class or individuals that will inform my next lesson?

Worksheets and answers

‘	Worksheet	Answers
For use with *Lesson 1*		
A: Basic Algebra of Complex Numbers	**x**	**x**
B: Simultaneous Linear Equation	**x**	**x**
For use with *Lesson 2*:		
C: Polynomial with Complex Roots	**x**	**x**
D: Square Roots of Complex Numbers	**x**	**x**
For use with *Lesson 3*:	**x**	**x**
E: Modulus and Argument	**x**	**x**
F: Product and Division in Polar Form	**x**	**x**
For use with Lesson plan 4:		
G: Locus in Argand Diagram	**x**	**x**

Worksheet A: Basic Algebra of Complex Numbers

1. Given the complex numbers $z_1 = 1 - 3\mathrm{i}$, $z_2 = 4 + \mathrm{i}$ and $z_3 = -2 + 3\mathrm{i}$, then find the following:

$$z_3^2 - 2z_1$$

2. Given the complex numbers $z_1 = 2 + \mathrm{i}$, $z_2 = 2 - 5\mathrm{i}$ and $z_3 = -1 + 2\mathrm{i}$, then find

$$\frac{2z_1 - 3z_2}{z_3}$$

3. Find $z \in \mathbb{C}$ that satisfies the equation $\frac{z+2}{1-\mathrm{i}} = \frac{z-3\mathrm{i}}{2+\mathrm{i}}$.

Worksheet B: Simultaneous Linear Equation

1. Solve the simultaneous equations for $z, w \in \mathbb{C}$

$$\frac{w}{z} = 3 - \mathrm{i}$$

$$z^* + 2w = 4 + 7\mathrm{i}$$

2. Solve the simultaneous equations for $w, v \in \mathbb{C}$

$$v + \mathrm{i}w = 4 - 2\mathrm{i}$$

$$(1 + 3\mathrm{i})v - w = 5 + \mathrm{i}$$

Worksheet C: Polynomial with Complex Roots

1. Solve the quadratic equation $z^2 + 4z + 13 = 0$

2. Given that $z = 1 + \mathrm{i}$ is a root of $z^4 + 3z^2 - 6z + 10 = 0$. Hence, find all the roots of this equation.

Worksheet D: Square Roots of Complex Numbers

1. Find the square root of the complex number $-3 + 4\mathrm{i}$

2. Find the square root of the complex number $7 - 24\mathrm{i}$

Worksheet E: Modulus and Argument

Show the full procedures:

1. Find the modulus and argument for the following complex numbers.
    a. $-12-5\mathrm{i}$
    b. $\sqrt{3}-\mathrm{i}$
    c. $-2+2\mathrm{i}$

2. Draw $2\ \mathrm{cis}\left(-\frac{\pi}{4}\right)$ and $3\ \mathrm{cis}\left(-\frac{5\pi}{6}\right)$ in Argand diagram and convert them into Cartesian form $z=a+b\mathrm{i}$

Worksheet F: Product and Division in Polar Form

1. Interpret the geometric meaning when a complex number is divided by $w = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$.

2. By expressing $2 - 2\sqrt{3}\mathrm{i}$ in the form of $re^{\mathrm{i}\theta}$ first, and hence find $\left(2 - 2\sqrt{3}\mathrm{i}\right)^5$.

Worksheet G: Locus in Argand Diagram

Question 1

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying both the inequalities $|z - 3\mathrm{i}| \leq 1$ and $\mathrm{Re}\, z \leq 0$, where $\mathrm{Re}\, z$ denotes the real part of z.

Find the greatest value of $\arg z$ for points in this region, giving your answer in radians correct to 2 decimal places. (2018 w P32 Q9)

Question 2

On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $-\frac{1}{3}\pi \leq \arg(z - 1 - 2\mathrm{i}) \leq \frac{1}{3}\pi$ and $\mathrm{Re}\, z \leq 3$, where $\mathrm{Re}\, z$ denotes the real part of z.

Calculate the least value of $\arg z$ for points in the region. Give your answer in radians

correct to 3 decimal places. (2023 m P32 Q2)

Worksheet A: Answers

1. Given the complex numbers $z_1 = 1 - 3\text{i}$, $z_2 = 4 + \text{i}$ and $z_3 = -2 + 3\text{i}$, then find the following:

$$z_3^2 - 2z_1$$

$$= (-2 + 3\text{i})^2 - 2(1 - 3\text{i})$$

$$= 4 + 9\text{i}^2 - 12\text{i} - 2 + 6\text{i}$$

$$= 4 - 9 - 2 - 6\text{i}$$

$$= -7 - 6\text{i}$$

2. Given the complex numbers $z_1 = 2 + \text{i}$, $z_2 = 2 - 5\text{i}$ and $z_3 = -1 + 2\text{i}$, then find

$$\frac{2z_1 - 3z_2}{z_3}$$

$$= \frac{2(2 + \text{i}) - 3(2 - 5\text{i})}{-1 + 2\text{i}}$$

$$= \frac{-2 + 17\text{i}}{-1 + 2\text{i}}$$

$$= \frac{(-2 + 17\text{i})(-1 - 2\text{i})}{(-1 + 2\text{i})(-1 - 2\text{i})}$$

$$= \frac{2 + 4\text{i} - 17\text{i} - 34\text{i}^2}{5}$$

$$= \frac{36 - 13\text{i}}{5}$$

3. Find $z \in \mathbb{C}$ that satisfies the equation $\frac{z+2}{1-\text{i}} = \frac{z-3\text{i}}{2+\text{i}}$.

$$(z + 2)(2 + \text{i}) = (1 - i)(z - 3\text{i})$$

$$2z + \text{i}z + 4 + 2\text{i} = z - 3\text{i} - \text{i}z + 3\text{i}^2$$

$$2z + \text{i}z - z + \text{i}z = -3\text{i} - 3 - 4 - 2\text{i}$$

$$z + 2z\text{i} = -7 - 5\text{i}$$

$$(1 + 2\text{i})z = -7 - 5\text{i}$$

$$z = \frac{-7 - 5\text{i}}{1 + 2\text{i}} = \frac{(-7 - 5\text{i})(1 - 2\text{i})}{(1 + 2\text{i})(1 - 2\text{i})}$$

$$z = \frac{-7 + 14\text{i} - 5\text{i} + 10\text{i}^2}{5}$$

$$z = \frac{-17 + 9\text{i}}{5}$$

Worksheet B: Answers

1. Solve the simultaneous equations for $z, w \in \mathbb{C}$

$$\frac{w}{z} = 3 - \mathrm{i}$$

$$z^* + 2w = 4 + 7\mathrm{i}$$

Let $z = a + b\mathrm{i}$, $w = (3 - \mathrm{i})(a + b\mathrm{i}) = 3a - \mathrm{i}a + 3b\mathrm{i} - b\mathrm{i}^2 = (3a + b) + (3b - a)\mathrm{i}$

$$z^* = a - b\mathrm{i}$$

$$a - b\mathrm{i} + 2[(3a + b) + (3b - a)\mathrm{i}] = 4 + 7\mathrm{i}$$

$$a - b\mathrm{i} + (6a + 2b) + (6b - 2a)\mathrm{i} = 4 + 7\mathrm{i}$$

$$(7a + 2b) + (5b - 2a)\mathrm{i} = 4 + 7\mathrm{i}$$

Therefore,

$$\begin{cases} 7a + 2b = 4 \\ 5b - 2a = 7 \end{cases}$$

$$a = \frac{2}{13}, b = \frac{19}{13}$$

2. Solve the simultaneous equations for $w, v \in \mathbb{C}$

$$v + \mathrm{i}w = 4 - 2\mathrm{i}$$

$$(1 + 3\mathrm{i})v - w = 5 + \mathrm{i}$$

Multiply equation 1 with i: $\mathrm{i}v - w = 4\mathrm{i} + 2$, this is equation 3

Use equation 2 minus equation 3: $(1 + 2\mathrm{i})v = 3 - 3\mathrm{i}$

$$v = \frac{3 - 3\mathrm{i}}{1 + 2\mathrm{i}} = -\frac{3}{5} - \frac{9}{5}\mathrm{i}$$

$$-\frac{3}{5} - \frac{9}{5}\mathrm{i} + \mathrm{i}w = 4 - 2\mathrm{i}$$

$$\mathrm{i}w = \frac{23}{5} - \frac{1}{5}\mathrm{i}$$

$$w = -\frac{1}{5} - \frac{23}{5}\mathrm{i}$$

Worksheet C: Answers

1. Solve the quadratic equation $z^2 + 4z + 13 = 0$

$$z = \frac{-4 \pm \sqrt{4^2 - 4 \times 13}}{2} = \frac{-4 \pm \sqrt{-36}}{2}$$

$$z = \frac{-4 \pm 6\mathrm{i}}{2} = -2 \pm 3\mathrm{i}$$

$$z_1 = -2 + 3\mathrm{i} \text{ and } z_2 = -2 - 3\mathrm{i}$$

2. Given that $z = 1 + \mathrm{i}$ is a root of $z^4 + 3z^2 - 6z + 10 = 0$. Hence, find all the roots of this equation.

According to the conjugate pair theorem, $z = 1 - \mathrm{i}$ is also going to be a root.

Therefore, $[z - (1 + \mathrm{i})][z - (1 - \mathrm{i})] = z^2 - 2z + 2$ is going to be a factor of $z^4 + 3z^2 - 6z + 10 = 0$.

By long division:

$$z^4 + 3z^2 - 6z + 10 = (z^2 - 2z + 2)(z^2 + 2z + 5) = 0$$

Therefore, $z^2 + 2z + 5 = 0$

$$z = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4\mathrm{i}}{2} = -1 \pm 2\mathrm{i}$$

$$z_1 = 1 + \mathrm{i}, z_2 = 1 - \mathrm{i}, z_3 = -1 + 2\mathrm{i}, z_4 = -1 - 2\mathrm{i}$$

Worksheet D: Answers

1. Find the square root of the complex number $-3 + 4\mathrm{i}$

$$(x + y\mathrm{i})^2 = x^2 - y^2 + 2xy\mathrm{i} = -3 + 4\mathrm{i}$$

$$\begin{cases} x^2 - y^2 = -3 \\ 2xy = 4 \end{cases}$$

By observation:

$$\begin{cases} x = 1 \\ y = 2 \end{cases} or \begin{cases} x = -1 \\ y = -2 \end{cases}$$

$$1 + 2\mathrm{i}\ or - 1 - 2\mathrm{i}$$

2. Find the square root of the complex number $7 - 24\mathrm{i}$

$$(x + y\mathrm{i})^2 = x^2 - y^2 + 2xy\mathrm{i} = 7 - 24\mathrm{i}$$

$$\begin{cases} x^2 - y^2 = 7 \\ 2xy = -24 \end{cases}$$

By observation:

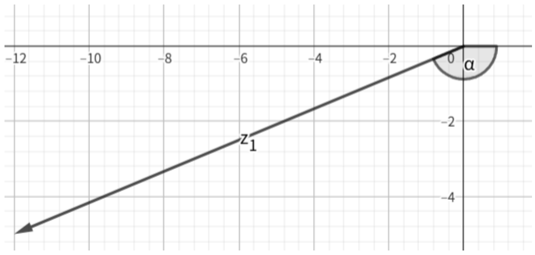
$$\begin{cases} x = 4 \\ y = -3 \end{cases} or \begin{cases} x = -4 \\ y = 3 \end{cases}$$

$$4 - 3\mathrm{i}\ or - 4 + 3\mathrm{i}$$

Worksheet E: Answers

1. Find the modulus and argument for the following complex numbers.
   a. $-12 - 5\mathrm{i}$
   b. $\sqrt{3} - \mathrm{i}$
   c. $-2 + 2\mathrm{i}$

$-12 - 5\mathrm{i}$ is in quadrant 3.

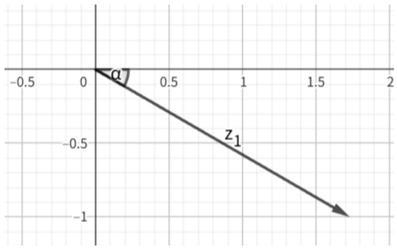


$$r = \sqrt{(-12)^2 + (-5)^2} = 13$$

$$\arg z = \arctan\left(-\frac{5}{-12}\right) \approx 0.395$$

By modification, $\arg z = 0.395 - \pi \approx -2.75$

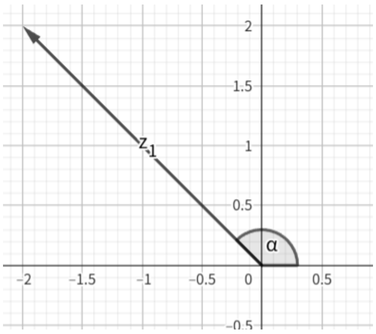
$\sqrt{3} - \mathrm{i}$ is in quadrant 4.



$$r = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$$

$$\arg z = \arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

$-2 + 2\mathrm{i}$ is in quadrant 2.

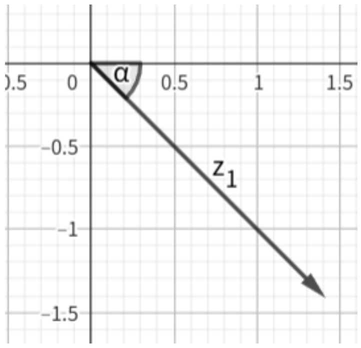


$$r = \sqrt{(-2)^2 + (2)^2} = 2\sqrt{2}$$

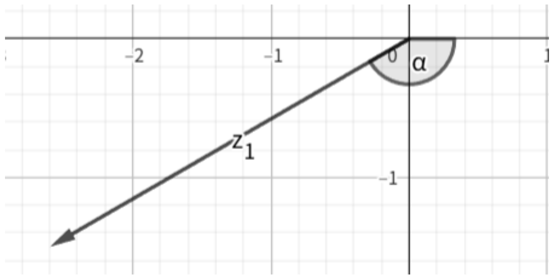
$$\arg z = \arctan\left(\frac{2}{-2}\right) = -\frac{\pi}{4}$$

By modification, $\arg z = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

2. Draw $2\text{ cis}\left(-\frac{\pi}{4}\right)$ and $3\text{ cis}\left(-\frac{5\pi}{6}\right)$ in Argand diagram and convert them into Cartesian form $z = a + bi$



$$2\text{ cis}\left(-\frac{\pi}{4}\right) = 2\cos\left(-\frac{\pi}{4}\right) + 2\text{i}\sin\left(-\frac{\pi}{4}\right) = 2 \times \frac{\sqrt{2}}{2} + 2\text{i}\left(-\frac{\sqrt{2}}{2}\right)$$
$$= \sqrt{2} - \sqrt{2}\text{i}$$



$$3\text{ cis}\left(-\frac{5\pi}{6}\right) = 3\cos\left(-\frac{5\pi}{6}\right) + 3\text{i}\sin\left(-\frac{5\pi}{6}\right)$$
$$= 3 \times \left(-\frac{\sqrt{3}}{2}\right) + 3\text{i}\left(-\frac{1}{2}\right) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}$$

Worksheet F: Answers

1. Interpret the geometric meaning when a complex number is divided by $w = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$.

$$w = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\arg w = \arctan\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$$

$$w = 1 \times e^{\mathrm{i}\left(\frac{\pi}{3}\right)} = e^{\mathrm{i}\left(\frac{\pi}{3}\right)}$$

Therefore, any complex number $z = re^{\mathrm{i}\theta}$ is divided by $w = e^{\mathrm{i}\left(\frac{\pi}{3}\right)}$,

$$\frac{z}{w} = \frac{re^{\mathrm{i}\theta}}{e^{\mathrm{i}\left(\frac{\pi}{3}\right)}} = re^{\mathrm{i}\left(\theta - \frac{\pi}{3}\right)}$$

The modulus of z does not change but the argument of z will decrease by $\frac{\pi}{3}$. (Rotation clockwise for $\frac{\pi}{3}$)

2. By expressing $2 - 2\sqrt{3}\mathrm{i}$ in the form of $re^{\mathrm{i}\theta}$ first, and hence find $\left(2 - 2\sqrt{3}\mathrm{i}\right)^5$.

$$r = \sqrt{(2)^2 + \left(-2\sqrt{3}\right)^2} = 4$$

$$\arg z = \arctan\left(\frac{-2\sqrt{3}}{2}\right) = \arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

$$2 - 2\sqrt{3}\mathrm{i} = 4e^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$$

$$\left(2 - 2\sqrt{3}\mathrm{i}\right)^5 = \left(4e^{\mathrm{i}\left(-\frac{\pi}{3}\right)}\right)^5 = 4^5 e^{\mathrm{i}\left(-\frac{5\pi}{3}\right)} = 1024e^{\mathrm{i}\left(\frac{\pi}{3}\right)}$$

$$= 1024\cos\left(\frac{\pi}{3}\right) + 1024\mathrm{i}\sin\left(\frac{\pi}{3}\right)$$

$$= 512 + 512\sqrt{3}\mathrm{i}$$

Worksheet G: Answers

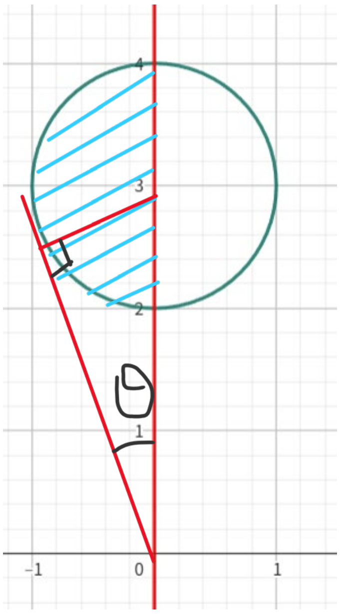
Question 1

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying both the inequalities $|z - 3\text{i}| \leq 1$ and $\text{Re}\, z \leq 0$, where $\text{Re}\, z$ denotes the real part of z.

Find the greatest value of $\arg z$ for points in this region, giving your answer in radians correct to 2 decimal places. (2018 w P32 Q9)

$|z - 3\text{i}| \leq 1$ means a circle center at $(0,3)$ with radius 1.

$\text{Re}\, z \leq 0$ means $x \leq 0$



$$\sin\theta = \frac{1}{3}$$

$$\theta \approx 0.3398$$

Maximum argument is $0.3398 + \frac{\pi}{2} \approx 1.91$

Question 2

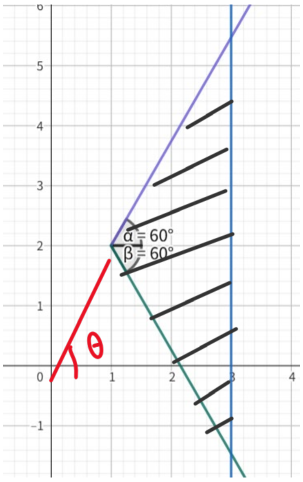
On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $-\frac{1}{3}\pi \leq \arg(z - 1 - 2\text{i}) \leq \frac{1}{3}\pi$ and $\text{Re}\, z \leq 3$, where $\text{Re}\, z$ denotes the real part of z.

Calculate the least value of $\arg z$ for points in the region. Give your answer in radians

correct to 3 decimal places. (2023 m P32 Q2)

$-\frac{1}{3}\pi \leq \arg(z - 1 - 2\mathrm{i}) \leq \frac{1}{3}\pi$ means the argument with respect to $(1,2)$

$\mathrm{Re}\, z \leq 3$ means $x \leq 3$



$$\tan\theta = \frac{2}{1}$$

$$\theta \approx 1.11$$

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