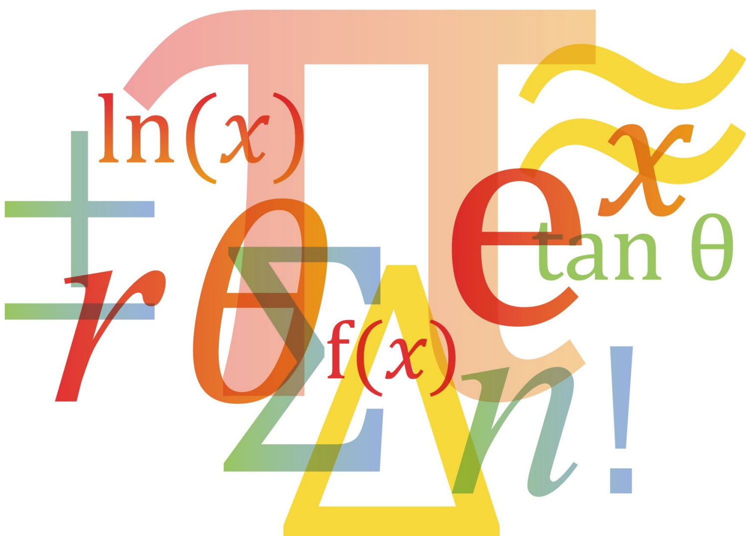
Cambridge Assessment International Education

Teaching Pack

2.3 Trigonometry

Cambridge International AS & A Level Mathematics 9709



Version 1

Cambridge Pathway

In order to help us develop the highest quality resources, we are undertaking a continuous programme of review; not only to measure the success of our resources but also to highlight areas for improvement and to identify new development needs.

We invite you to complete our survey by visiting the website below. Your comments on the quality and relevance of our resources are very important to us.

www.surveymonkey.co.uk/r/GL6ZNJB

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Contents

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Introduction

This pack will help you to develop your learners' skills in mathematical thinking and mathematical communication, which are essential for success at AS & A Level and in further education.

Mathematical thinking and communication will be developed by focussing on:

1. Conceptual understanding – the 'why' behind the 'what'
2. Strategic competence – forming and solving problems
3. Adaptive reasoning – explanations, justifications and deductive reasoning

Throughout all activities, the learners will also develop:

- Procedural fluency – know when, how and which rules to use
- Positive disposition – believe maths can be learned, applied and is useful
- Their skills in writing mathematically – writing working & proofs

These link to the course Assessment Objectives (AOs) which you can find in detail in the syllabus:

A01 Knowledge and understanding

A02 Application and communication

Each *Teaching Pack* contains one or more lesson plans and associated resources, complete with a section of preparation and reflection.

Each lesson is designed to be an hour long but you should adjust the timings to suit the lesson length available to you and the needs of your learners.

Important note
Our *Teaching Packs* have been written by **classroom teachers** to help you deliver topics (but not necessarily a whole topic) and skills that can be challenging. Use these materials to supplement your teaching and engage your learners. You can also use them to help you create lesson plans for other topics.

This content is designed to give you and your learners the chance to explore a more active way of engaging with mathematics that encourages independent thinking and a deeper conceptual understanding. It is not intended as specific practice for the examination papers.

The *Teaching Packs* are designed to provide you with some example lessons of how you might deliver content. You should adapt them as appropriate for your learners and your centre. A single pack will only contain at most five lessons, it will **not** cover a whole topic. You should use the lesson plans and advice provided in this pack to help you plan the remaining lessons of the topic yourself.

Lesson preparation

This *Teaching Pack* will cover the following syllabus content.

Candidate should be able to:	Notes and examples
• understand the relationship of the secant, cosecant and cotangent functions to cosine, sine and tangent	
• use trigonometrical identities for the simplification and exact evaluation of expressions, and in the course of solving equations and select an identity or identities appropriate to the context, showing familiarity in particular with the use of — $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$ — the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$ — the formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$	e.g. simplifying $\cos(x - 30°) - 3\sin(x - 30°)$. e.g. solving $\tan\theta + \cot\theta = 4$, $2\sec^2\theta - \tan\theta = 5$, $3\cos\theta + 2\sin\theta = 1$.

The remaining two partial bullet points for topic 2.3 Trigonometry are not covered in this Teaching Pack (see the syllabus for detail). You will need to write your own lesson plans for these items.

Candidate should be able to:	Notes and examples
• use properties and graphs of all six trigonometric functions for angles of any magnitude • understand the relationship between a graph and the expression of $a\sin\theta + b\cos\theta$ in the forms $R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$	

Dependencies

For all lesson plans in this Teaching Pack, knowledge from the following 9709 syllabus content is required.

Candidate should be able to:	Notes and examples
• recognise and solve equations in x which are quadratic in some function of x	e.g. $x^4 - 5x^2 + 4 = 0$, $6x + \sqrt{x} - 1 = 0$, $\tan^2 x = 1 + \tan x$
• understand the definition of a radian	
• use the exact values of the sine, cosine and tangent of 30°, 45°, 60°, and related angles	e.g. $\cos 150° = -\frac{1}{2}\sqrt{3}$, $\sin \frac{3}{4}\pi = \frac{1}{\sqrt{2}}$
• use the identities $\frac{\sin\theta}{\cos\theta} \equiv \tan\theta$ and $\sin^2\theta + \cos^2\theta \equiv 1$	e.g. in proving identities, simplifying expressions and solving equations
• find all the solutions of simple trigonometrical equations lying in a specified interval (general forms of solution are not included).	e.g. solve $3 \sin 2x + 1 = 0$ for $-\pi < x < \pi$, $3 \sin^2\theta - 5 \cos\theta - 1 = 0$ for $0° \le \theta \le 360°$.

Prior knowledge and skills

For all lessons, it is assumed that learners have already completed Cambridge IGCSE™ Mathematics 0580, or a course at an equivalent level. See the syllabus for more details of the expected prior knowledge for taking Cambridge International AS & A Level Mathematics 9709.

When planning any lesson, make a habit of always asking yourself the following questions about your learners' prior knowledge and skills:

- Do I need to re-teach this or do learners just need some practice?
- Is there an interesting activity that will efficiently achieve this?

Learners should be able to rearrange equations to be able to solve these, and should be familiar in solving trigonometric equations to find all the solutions within a given interval. Although specific dependencies are mentioned for each lesson in the Teaching Pack, note also that all the knowledge of the content for Paper 1: Pure Mathematics 1 is assumed, and learners may be required to demonstrate such knowledge in answering questions.

Key learning objectives

The following list represents the main underlying concepts that you should make sure your learners have understood by the end of this topic.

- When solving trigonometric equations finding all solutions by, where appropriate, factorising, taking both positive and negative square roots and ensuring correct accuracy is used
- When proving trigonometric identities ensuring all relevant steps are included

Why this topic matters

Trigonometry is used in resolving forces in mechanics, working with projectile motion and components of vectors in addition to being used in the differentiation and integration components of the course.

Key terminology and notation

Your learners will need to be confident with the following terminology and notation.

equation: $2\sec^2\theta - \tan\theta = 5$

identity: $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$

prove: showing a statement is true using definitions

solve: finding all possible answers to an equation

radians: a unit of measure of angles, where one radian is the angle formed at the centre of a circle where the arc formed is the same length as the radius of the circle

Insights video

There is an Insights video linked to this *Teaching Pack*:

- **2.3 Trigonometry** – watch this video which will show some of the misunderstandings learners have when solving questions with trigonometric quadratic, or cubic, equations.

Teacher tutorials

There are two tutorials linked to this *Teaching Pack*:

- **Solving Trigonometric Equations**
  Review this tutorial before teaching Lesson Plan 1; this will help you solve a trigonometric equation where it is important to factorise an expression rather than cancel terms, and also looks at the importance of giving answers to a required degree of accuracy.
- **Trigonometric Identities**
  Review this tutorial before teaching Lesson Plan 3; this will help you make sure all steps are included when proving a trigonometric identity.

Lesson progression

Lesson 1 looks at solving simple trigonometric equations which involve square roots and factorising. This lesson also covers accuracy when working in degrees or radians.
Lesson 2 covers solving trigonometric equations involving multiple or fractional angles.
Lesson 3 looks at proving trigonometric identities, using formulae learners should already be familiar with, deciding on which side to begin working and looking at what is required in a rigorous proof to gain full marks.

Going forward

This topic links to differentiation in topics 2.4 and 3.4; integration in topics 2.5 and 3.5; and further trigonometry in topic 3.3.

Lesson 1: Solving Trigonometric Equations

Preparation	• Review the Teacher tutorial: *Solving Trigonometric equations* Pay particular attention to the notes for each slide. • Practice sketching trigonometric graphs on a graphical calculator in both degrees and radians, particularly for multiple and fractional angles of trigonometric functions to build confidence in sketching these without the use of a graphical calculator
Resources	• Paper, Mini whiteboards or other writing materials • Lesson slides: *Solving Trigonometric equations* • Worksheets A to C • Graphical calculators
Learning objectives	By the end of the lesson: • ***all*** learners should find a solution when solving a trigonometric equation • ***most*** learners should find most solutions when solving a trigonometric equation, although some may be missing when equations become more complicated • ***some*** learners should find all solutions when solving a trigonometric equation, including where factorising and taking square roots are needed

Dependencies

Learners should already be familiar with the identities
$\frac{\sin\theta}{\cos\theta} \equiv \tan\theta$, $\sin^2\theta + \cos^2\theta \equiv 1$, $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$,

and the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$.

Common misconceptions

Misconception	Problems this can cause	An example way to resolve the misconception
When solving equations such as quadratics or cubics, common factors involving the variable can be cancelled. When taking square roots, only the positive answer is found.	Solutions are missing, resulting in a loss of marks.	Questions 1 and 2 from the Starter below; also the Teacher tutorial Solving Trigonometric Equations.

Timings	Activity
15 min	**Starter/Introduction** Issue Worksheet A: Starter and ask all learners to think about the four questions, and attempt these on their own. Then pair with another learner to share answers. Finally, collate answers found as a group to ensure all answers have been found for each question. You may wish to pair learners according to ability, with one more able and

Timings	Activity
	one less able learner in each pair. Q1 and 2 test whether learners are dividing through by common factors and therefore losing solutions. Q2 also tests whether both positive and negative square roots are found. Q3 checks for finding all solutions in a given range in degrees, and whether answers have been given correctly to 1 decimal place (as working in degrees). Q4 checks for finding all solutions in a given range in radians, and whether answers have been given correctly to 3 significant figures (as working in radians). Demonstrate using a graphical calculator (you may also wish to encourage the use of graphical calculators with learners) how the graphs of $\sin x$ (Q3) and $\cos x$ (Q4) can show how many solutions to expect, and provide a check to those answers calculated algebraically. Note: graphical calculators are a useful teaching aide, but they are not permitted within the examination. Also learners need to be aware that answers found without working will not be acceptable in an examination. Depending upon the outcome of the starter, you may wish to spend more time solving simple equations before moving on to trigonometric equations, or you may find more work using the CAST system or trigonometric graphs is needed to ensure learners can find all possible solutions (in radians or degrees) in a given range.
15 min 25 min	**Main lesson** Show the Lesson slides: Solving Trigonometric Equations. Encourage learners to state each step of the solution. Check to see whether any learners suggest cancelling cot θ at any stage of the working, and point out the loss of solutions if this done, linking to Q1 and Q2 in the starter. Encourage full and clear working. Use the notes on the PowerPoint to stress what is required at each stage. Ensure the final solutions are given to 1 decimal place, as we are working in degrees, linking to Q3 in the starter. Also note there should be no additional solutions given, even those which are outside the range requested in the question. Issue Worksheet B: Trigonometry Formula Sheet and Worksheet C: Solving trigonometric equations, and ask learners to attempt to solve each equation for the given range. Encourage use of Worksheet B to help in their working. Note the accuracy shown on the answer sheet for full marks to be awarded in each case. There may be different valid approaches used in each question, but the same solutions should be found in each case. You may wish to use some of this worksheet as a homework exercise to mark yourself.
5 min	**Plenary** What have you learned in today's lesson? Explain to your partner one thing you can now use that you didn't know (or had forgotten) before the lesson.

Reflection	Reflect on your lesson, use the **Lesson reflection** notes to help you.
Are all learners able to confidently find all/most solutions in a given range? Is it necessary to produce more work looking at trigonometric graphs or the CAST system?	

Lesson plan 2: Solving Trigonometric Equations with Multiple or Fractional Angles

Preparation	• Practice sketching trigonometric graphs on a graphical calculator in both degrees and radians, particularly for multiple and fractional angles of trigonometric functions, as in lesson 1 to build confidence in sketching these without the use of a graphical calculator

Resources	• Paper, Mini whiteboards or other writing materials • Worksheets B and D • Graphical calculators

Learning objectives	By the end of the lesson: • ***all*** learners should find a solution when solving a trigonometric equation with multiple or fractional angles • ***most*** learners should find most solutions when solving a trigonometric equation with multiple or fractional angles, although some may be missing or additional incorrect solutions may be found • ***some*** learners should find all solutions when solving a trigonometric equation with multiple or fractional angles with no additional incorrect solutions

Dependencies

Learners should already be familiar with the identities
$\frac{\sin\theta}{\cos\theta} \equiv \tan\theta$, $\sin^2\theta + \cos^2\theta \equiv 1$, $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$,

the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$,

and the formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$.

Common misconceptions

Misconception	Problems this can cause	An example way to resolve the misconception
When solving trigonometric equations with multiple or fractional angles, thinking there are always the same number of solutions to be found for each question.	Solutions are missing, resulting in a loss of marks.	The example in the main lesson below, or sketching trigonometric graphs to consider how many solutions there should be for a given function in a given range.

Timings	Activity
15 min	**Starter/Introduction** Ask each learner to find an equation which would give solutions of 15º, 75º, 195º, 255º for the range $0° \leq x \leq 360°$. **Challenge:** To find four different equations to produce the given solutions. **Support:** Ask learners to sketch a graph of $y = \sin 2x$ and find where the solutions appear on the graph. $\sin 2x = 0.5$ is one possible solution; there are many others including $\sin x \cos x = 0.25$. Other more complex solutions could include multiples of other functions which produce no solutions; for example $(2\sin 2x - 1)(\sin 2x + 2) = 0$ which could be written in the form $2\sin^2 2x + 3\sin 2x - 2 = 0$ or even $2\cos^2 2x - 3\sin 2x = 0$. One thing to note is that as there are four solutions in the range $0° \leq x \leq 360°$, that unless working with a tangent function, the equation is likely to be a function of $\sin 2x$ or $\cos 2x$.
15 min 20 min	**Main lesson** Using the equation $2\cos^2 2x - 3\sin 2x = 0$ as an example, demonstrate how this could be solved for the range $0° \leq x \leq 360°$. Firstly ask learners to investigate, if necessary with the help of a graphical calculator, how many solutions should be expected within the given range. Learners can also predict the answers, but as in Lesson 1, note that graphical calculators are a useful teaching aide, but they are not permitted within the examination. Also learners need to be aware that answers found without working will not be acceptable in an examination. Note there are four solutions to be found, and that the answers are 15º, 75º, 195º, 255º; the same as those issued in the starter question. Using the same technique as at the start of the main lesson, use Worksheet B: Trigonometry Formula Sheet from Lesson 1 and issue Worksheet D: Multiples and Fractions and, working in (mixed ability, where appropriate) groups, give each group one question to research from the worksheet, using questions 1, 2, 3, 4(b) and 6. Learners should firstly attempt to identify how many solutions to expect by sketching the graph, and then work together using Worksheet B to find those solutions showing all working. Then bring the class together, and ask each group to explain to the rest of the class how they constructed their solutions. **Challenge:** Q4(a) and Q5 from Worksheet D
10 min	**Plenary** How many solutions should the equation $\cos 3x = \frac{\sqrt{3}}{2}$ have, in the range $0° \leq x \leq 720°$?

Timings	Activity
	Challenge: Find all the solutions. **Support:** Begin by sketching the graph $y = \cos 3x$ for $0° \leq x \leq 720°$. If more support is required, this could be done with the use of a graphical calculator. Note: There are 12 solutions: $x = 10°, 110°, 130°, 230°, 250°, 350°, 370°, 470°, 490°, 590°, 610°, \text{and } 710°$.

Reflection	Reflect on your lesson, use the **Lesson reflection** notes to help you.
Are all learners able to confidently find all/most solutions in a given range without the use of a graphical calculator? Do trigonometric graph of multiple/fractional angles need more work?	

Lesson plan 3: Proving Trigonometric Identities

Preparation	• Review the Teacher tutorial *Trigonometric Identities*. Pay particular attention to the notes for each slide.
Resources	• Worksheets B and E to H • Lesson slides: *Trigonometric Identities*
Learning objectives	By the end of the lesson: • ***all*** learners should identify which side of the identity will be the better option to start with • ***most*** learners should prove identities although some steps may be missing • ***some*** learners should successfully prove identities, stating explicitly all those trigonometric identities which have been used.

Dependencies

Learners should already be familiar with the identities

$\frac{\sin\theta}{\cos\theta} \equiv \tan\theta$, $\sin^2\theta + \cos^2\theta \equiv 1$, $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$,

the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$,

and the formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$.

Common misconceptions

Misconception	Problems this can cause	An example way to resolve the misconception
That all steps do not need to be explicitly stated in order to gain full marks in a question, but that reaching one side of the identity from the other is enough for full marks.	Marks needlessly lost through assumptions.	The example used in Worksheet F together with the mark scheme contained in the answers should help to test whether learners are showing full working. This mark scheme should also help learners to see what is required in a solution to gain full marks.

Timings	Activity
5 min	**Starter/Introduction** Use Worksheet E: Which side? to decide, in pairs, on which side of the identity the work should begin. Base the decision upon which side of the identity appears the more complicated, and therefore gives more option to simplify. The identities do not need to be proved at this stage; this will come later in the lesson. Note Q1 is the identity proved in the teacher tutorial.
10 min	**Main lesson** Issue Worksheet B: Trigonometry Formula Sheet and Worksheet F: State the obvious! to each learner and ask them to try, individually, the first bullet point on Worksheet F.

Timings	Activity
	Use Worksheet B to help. After 2 minutes, issue Worksheet F: Answers and ask learners to swap their answers. They should mark the answer in accordance with the guidance on Worksheet F: Answers. The idea of this activity is to highlight to all learners the cost of not showing all working in a proof, no matter how obvious it may seem. **Support:** A hint for less able learners could be to begin on the left side of the identity. If more help is needed, they could be told to begin by factorising the left side. There may be some discussion at this stage as to whether a response is worth 1 or 2 marks. Encourage the markers to query or explain how the marks should be awarded.
15 min	Show the Lesson slides: Trigonometric Identities. Note this is Q1 on Worksheet E. Encourage learners to state each step of the solution, with the help of Worksheet B. Encourage full and clear working, including stating which trigonometric identities are used at each stage. Remember the aim is not just to arrive at the opposite side of the trigonometric statement, but to do so in a clear and mathematically rigorous way. Use the notes on the PowerPoint to stress what is required at each stage.
25 min	Issue Worksheet G: Show that… and ask learners to attempt to provide a rigorous argument to show the statement is true. Encourage use of Worksheet B to help in their working. Note the steps provided on the answer sheet for full marks to be awarded in each case. There may be different valid approaches used in each question, but a similar step should be seen for each attempt. You may wish to use some of this worksheet as a homework exercise to mark yourself.
	Plenary Issue Worksheet H: Assess the attempt and ask learners to individually improve upon the solution provided, using what they have learned during the lesson. Compare their results with a partner, and try to improve the solution further. Go through as a class and ask each pair how the solution could be improved.

Reflection	Reflect on your lesson, use the **Lesson reflection** notes to help you.

Planning your own lessons

You now need to plan lessons to cover the following bullet points:

Candidate should be able to:	Notes and examples
• use properties and graphs of all six trigonometric functions for angles of any magnitude • understand the relationship between a graph and the expression of $a\sin\theta + b\cos\theta$ in the forms $R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$	

Follow the structure of the *Teaching Pack*, and use techniques from the 'How to' guides, to create your own engaging lessons to cover these bullet points. Consider what preparation you need for each lesson: what prior knowledge is needed, what are the key objectives, what are the dependencies, what common misconceptions are there, and so on.

Below, we have provided an outline of some activities and approaches you might like to try.

Lesson 4: Using graphs of all six trigonometric functions for angles of any magnitude
Common misconceptions: each graph covers only one period
Starter: You could try discussing differences and similarities between the graph $y = \sin x$ and the graph $y = 4 + 3\sin 2x$
Main: You could use resource Tangled Trig Graphs from the Nrich website https://nrich.maths.org/6481 and have a class discussion about how transformations affect the shape of a graph.
Plenary: You could try asking each learner to sketch their own graph, then swap with a neighbour and ask them to identify the function of the graph drawn.

Lesson 5: Graphs and $a\sin\theta + b\cos\theta$
Common misconceptions: For the graph $y = 3\sin\theta + 4\cos\theta$. the graphs of $y = 3\sin\theta$ and $y = 4\cos\theta$ need to be drawn then 'added'.
Starter: You could try sketching the graph $y = 3\sin\theta + 4\cos\theta$ and the graph $y = 5\sin(\theta + 36.87°)$
Main: You could use resource Using Rcos(x + α) to Find the Maximum and Minimum Values of a Function and to Solve a Trigonometric Equation from the Stem website https://www.stem.org.uk/elibrary/resource/34837 and have a class discussion about how, for example, doubling the values of both 'a' and 'b' affects the values of R and α. Note the first video on the website solves the two problems mathematically as should be done by learners in examinations; the second and third videos demonstrate how the solutions already found can be seen graphically on a graphical calculator and can be ignored. Again, note: graphical calculators are a useful teaching aide, but they are not permitted within the examination. Also learners need to be aware that answers found without working will not be acceptable in an examination.
Plenary: You could try expressing one function in both forms $R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$

You will find some other activity suggestions in the Scheme of Work.

Lesson reflection

As soon as possible after the lesson you need to think about how well it went.

One of the key questions you should always ask yourself is:

Did all learners get to the point where they can access the next lesson? If not, what will I do?

Reflection is important so that you can plan your next lesson appropriately. If any misconceptions arose or any underlying concepts were missed, you might want to use this information to inform any adjustments you should make to the next lesson.

It is also helpful to reflect on your lesson for the next time you teach the same topic. If the timing was wrong or the activities did not fully occupy the learners this time, you might want to change some parts of the lesson next time. There is no need to re-plan a successful lesson every year, but it is always good to learn from experience and to incorporate improvements next time.

To help you reflect on your lesson, answer the most relevant questions below.

Were the lesson objectives realistic?
What did the learners learn today? Or did they learn what was intended? Why not?
What proportion of the time did we spend on the most important topics?
Were there any common misconceptions?
What do I need to address next lesson?
What was the learning atmosphere like?
Did my planned differentiation work well?
How could I have helped the lowest achieving learners to do more?
How could I have stretched the highest achieving learners even more?
Did I stick to timings?
What changes did I make from my plan and why?

Summary evaluation

What two things went really well? (Consider both teaching and learning.)

What two things would have improved the lesson? (Consider both teaching and learning.)

What have I learned from this lesson about the class or individuals that will inform my next lesson?

Worksheets and answers

Worksheet A: Starter

Solve

1) $x^2 = 2x$

2) $x^3 = 4x$

3) $\sin x = 0.6$ for $0° \leq x < 360°$

4) $\sec x = -2.5$ for $0 \leq x < 2\pi$

Worksheet B: Trigonometry Formula Sheet

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1, \qquad 1 + \tan^2\theta \equiv \sec^2\theta, \qquad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

Principal values:

$$-\tfrac{1}{2}\pi \leqslant \sin^{-1} x \leqslant \tfrac{1}{2}\pi, \qquad 0 \leqslant \cos^{-1} x \leqslant \pi, \qquad -\tfrac{1}{2}\pi < \tan^{-1} x < \tfrac{1}{2}\pi$$

Worksheet C: Solving trigonometric equations

Solve

1) $4\sin^2 x = 3$ for $0° \leq x < 360°$

2) $3\sec x = 2\cot x$ for $0 \leq x < 2\pi$

3) $4\sin^3 x - \sin x = 0$ for $0 \leq x < 2\pi$ giving exact values

4) $3\cot^2 x = -\cot x$ for $0° \leq x < 360°$

Worksheet D: Multiples and Fractions

1) Solve $2\cos 2\theta = 1 - 2\cos\theta$ for $0° \le x < 360°$

2) Solve $\frac{2\sin x}{\sin(x/2)} + 2\cos\left(\frac{x}{2}\right) = 3$ for $0° \le x < 360°$

3) Solve $4\sin 3x = 2$ for $0 \le x < 2\pi$ giving exact values

4) a) Prove $\tan x + \cot x \equiv 2\operatorname{cosec} 2x$
   b) Hence solve $\tan x + \cot x = 5$ for $0° \le x < 360°$

5) If $\tan x = \frac{\sqrt{3}}{2}$, find the exact value of $\cot 2x$ without the use of a calculator, showing all working.

6) Solve $\tan 2x = 3\tan x$ for $-180° \le x < 180°$

Worksheet E: Which side?

These identities are to be proved at a later stage. Decide, in pairs, on which side of the identity the work should begin (left or right). Base the decision upon which side of the identity appears the more complicated, and therefore gives more option to simplify.

1) $\cot(45 + A) + \cot(45 - A) \equiv 2\sec 2A$

2) $\frac{2}{\sin 2A} \equiv \tan A + \cot A$

3) $\frac{\cos 2A}{\sin^2 A} \equiv \operatorname{cosec}^2 A - 2$

4) $\frac{\cos 2A}{\cos^2 A} \equiv 2 - \sec^2 A$

5) $4\operatorname{cosec}^2 x \equiv \frac{\sec x}{\cos x} + \frac{\operatorname{cosec} x}{\sin x}$

6) $\cos 3x \equiv 4\cos^3 x - 3\cos x$

Worksheet F: State the obvious!

- Show that $\sin^3 x + \cos^2 x \sin x \equiv \sin x$

- Swap papers and mark using the mark scheme.

Worksheet G: Show that…

With the exception of 2(c), these identities appeared in Worksheet E. Use the answers to Worksheet E to decide on which side you should start your proof.

1) Show that $\frac{2}{\sin 2A} \equiv \tan A + \cot A$

2) (a) Show that $\frac{\cos 2A}{\sin^2 A} \equiv \operatorname{cosec}^2 A - 2$

(b) Show that $\frac{\cos 2A}{\cos^2 A} \equiv 2 - \sec^2 A$

(c) Hence show that $\frac{\cos 2A}{\sin^2 A} + \frac{\cos 2A}{\cos^2 A} \equiv \cot^2 A - \tan^2 A$

3) Show that $4 \operatorname{cosec}^2 x \equiv \frac{\sec x}{\cos x} + \frac{\operatorname{cosec} x}{\sin x}$

4) Show that $\cos 3x \equiv 4 \cos^3 x - 3 \cos x$

Worksheet H: Assess the attempt

This worksheet shows an attempt at answering the question:

Show that $\tan 2x \cot x \equiv \frac{2\cos^2 x}{\cos 2x}$

Here is the attempt:

$$\tan 2x \cot x \equiv \frac{\sin 2x}{\cos 2x}\cot x \equiv \frac{2\cos x \cos x}{\cos 2x} \equiv \frac{2\cos^2 x}{\cos 2x}$$

Use what you have learned in the lesson to show how the answer could have been clearer.

What steps has the candidate missed out?

Worksheet A: Answers

1) $x^2 - 2x = 0$

   $x(x - 2) = 0$

   $x = 0 \text{ or } 2$ (do not divide by x)

2) $x^3 - 4x = 0$

   $x(x^2 - 4) = 0$

   $x = 0, 2 \text{ or} - 2$ (do not divide by x; also ensure both positive and negative square roots are found)

3) $x = 36.9° \text{ or } 143.1°$ (note answers in degrees should be to 1 decimal place)

4) $\cos x = -0.4$ $x = 1.98 \text{ or } 4.30$ (note answers in radians should be to 3 significant figures)

Worksheet C: Answers

1) $\sin^2 x = \frac{3}{4}$ $\quad \sin x = \pm\frac{\sqrt{3}}{2}$

When $\sin x = +\frac{\sqrt{3}}{2}, x = 60°$ or $180 - 60 = 120°$

When $\sin x = -\frac{\sqrt{3}}{2}, x = -60°$ (not in range) so $x = -60 + 360 = 300°$ or $180 + 60 = 240°$

So $x = 60°, 120°, 240°$ or $300°$

2) $\frac{3}{\cos x} = \frac{2\cos x}{\sin x}$

$3\sin x = 2\cos^2 x$

$3\sin x = 2(1 - \sin^2 x)$

$3\sin x = 2 - 2\sin^2 x$

$2\sin^2 x + 3\sin x - 2 = 0$

$(2\sin x - 1)(\sin x - 2) = 0$

$\sin x = 0.5$ or $\sin x = 2$ (invalid)

When $\sin x = 0.5, x = 0.524$ or $x = \pi - 0.524 = 2.62$

So $x = 0.524^c$ or 2.62^c (note answers in radians should be to 3 significant figures)

3) $\sin x(4\sin^2 x - 1) = 0$

$\sin x(2\sin x + 1)(2\sin x - 1) = 0$

$\sin x = 0, -0.5$ or $+0.5$

When $\sin x = 0, x = 0$ or π

When $\sin x = -0.5, x = -\frac{\pi}{6}$ (not in range) so $x = -\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}$ or $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$

When $\sin x = +0.5, x = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

So $x = 0, \pi, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$ (note exact answers required)

4) $3\cot^2 x = -\cot x$

$3\cot^2 x + \cot x = 0$

$\cot x(3\cot x + 1) = 0$

$\cot x = 0, \cot x = -\frac{1}{3}$

When $\cot x = 0, \tan x \to \infty, x = 90$ *or* $90 + 180 = 270$

When $\cot x = -\frac{1}{3}, \tan x = -3, x = -71.56$ (out of range) so $x = -71.56 + 180 = 108.4$ or $180 + 108.4 = 288.4$

So $x = 90.0°, 270.0°, 108.4°$ or $288.4°$ (note answers in degrees should be to 1 decimal place)

Worksheet D: Answers

1) Using $\cos 2\theta = 2\cos^2\theta - 1$ the equation becomes
$2(2\cos^2\theta - 1) = 2\cos\theta - 1$
$4\cos^2\theta - 2\cos\theta - 1 = 0$
$\cos\theta = \frac{2\pm\sqrt{(-2)^2-4(4)(-1)}}{2(4)} = \frac{2\pm\sqrt{20)}}{8}$
When $\cos\theta = \frac{2+\sqrt{20)}}{8}, \theta = 36 \text{ or } 360 - 36 = 324$
When $\cos\theta = \frac{2-\sqrt{20)}}{8}, \theta = 108 \text{ or } 360 - 108 = 252$
So $\theta = 36°, 108°, 252° \text{ or } 324°$

2) Using $\sin 2\theta = 2\sin\theta\cos\theta$ where $\theta = \frac{x}{2}$ the equation becomes
$\frac{2[2\sin(x/2)\cos(x/2)]}{\sin(x/2)} + 2\cos\left(\frac{x}{2}\right) = 3$
$4\cos\left(\frac{x}{2}\right) + 2\cos\left(\frac{x}{2}\right) = 3$
$6\cos\left(\frac{x}{2}\right) = 3$
$\cos\left(\frac{x}{2}\right) = 0.5$
$\frac{x}{2} = 60 \text{ or } 360 - 60 = 300$
So $x = 120°$

3) As $4\sin 3x = 2$ then $\sin 3x = \frac{2}{4} = 0.5$
$3x = \frac{\pi}{6} \text{ or } \pi - \frac{\pi}{6} = \frac{5\pi}{6} \text{ or } 2\pi + \frac{\pi}{6} = \frac{13\pi}{6} \text{ or } 2\pi + \frac{5\pi}{6} = \frac{17\pi}{6} \text{ or } 4\pi + \frac{\pi}{6} = \frac{25\pi}{6} \text{ or } 4\pi + \frac{5\pi}{6} = \frac{29\pi}{6}$
So $x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18} \text{ or } \frac{29\pi}{18}$ (note exact answers required)

4) a) $\tan \boldsymbol{x} + \cot x \equiv 2\operatorname{cosec} 2x$ (stated in question)

Starting with the left side
$\tan \boldsymbol{x} + \cot x \equiv \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$

$\equiv \frac{\sin^2 \boldsymbol{x} + \cos^2 x}{\sin \boldsymbol{x} \cos x}$

$\equiv \frac{1}{\sin \boldsymbol{x} \cos x}$

$\equiv \frac{2}{2\sin \boldsymbol{x} \cos x}$

$\equiv \frac{2}{2\sin \boldsymbol{2x}}$

$\equiv 2\operatorname{cosec} 2x$ as required

b) Use the answer to a), but note if this cannot be proved in a) the result may still be used in b).
The word 'hence' is the clue to this.
The equation becomes $2\operatorname{cosec} 2x = 5$
$\frac{2}{\sin 2x} = 5$
$\sin 2x = \frac{2}{5} = 0.4$

Worksheet D: Answers continued

$2x = 23.57 \ldots \text{or } 180 - 23.57 \ldots = 156.42 \ldots \text{ or } 23.57 \ldots + 360 = 383.57 \ldots \text{or } 156.42 \ldots + 360 = 516.42 \ldots$

So $x = 11.8°, 78.2°, 191.8° \text{ or } 258.2°$ (note answers in degrees should be to 1 decimal place)

b)

5) $\tan 2x \equiv \frac{2\tan x}{1-\tan^2 x}$

As $\tan \boldsymbol{x} = \frac{\sqrt{3}}{2}$, $\tan 2x = \frac{2\left(\frac{\sqrt{3}}{2}\right)}{1-\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{\sqrt{3}}{1-\frac{3}{4}} = \frac{\sqrt{3}}{\frac{1}{4}} = 4\sqrt{3}$

So $\cot 2x = \frac{1}{\tan 2x} = \frac{1}{4\sqrt{3}} = \frac{1}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{12}$

6) Using $\tan 2x \equiv \frac{2\tan x}{1-\tan^2 x}$ the equation becomes $\frac{2\tan x}{1-\tan^2 x} = 3\tan x$

So $2\tan x = 3\tan x(1-\tan^2 x)$

$\tan x(3 - 3\tan^2 x - 2) = 0$

$\tan x(1 - 3\tan^2 x) = 0$

$\tan x = 0 \text{ or } \tan^2 x = \frac{1}{3} \text{ so } \tan x = \pm\frac{1}{\sqrt{3}}$

When $\tan x = 0, x = 0, -180$

When $\tan x = +\frac{1}{\sqrt{3}}, x = 30 \text{ or } 30 - 180 = -150$

When $\tan x = -\frac{1}{\sqrt{3}}, x = -30 \text{ or } -30 + 180 = 150$ So $x = 0°, -180°, -150°, -30°, 30° \text{ or } 150°$

Worksheet E: Answers

1) left
2) right
3) left
4) left
5) right
6) left

Worksheet F: Answers

$\sin^3 x + \cos^2 x \sin x \equiv \sin x$

Start with the left side and factorise correctly

$\sin^3 x + \cos^2 x \sin x \equiv \sin x(\sin^2 x + \cos^2 x)$ 1 mark

State $\sin^2 x + \cos^2 x \equiv 1$ OR clearly replace $\sin^2 x + \cos^2 x$ with the value 1
AND show the expression becomes $\sin x$

$\sin x(\sin^2 x + \cos^2 x) \equiv \sin x \times 1 \equiv \sin x$ 1 mark

Worksheet G: Answers

1) $\frac{2}{\sin 2A} \equiv \tan A + \cot A$

Start with the right side

$$\tan A + \cot A \equiv \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$$

$$\equiv \frac{\sin^2 A + \cos^2 A}{\cos A \sin A}$$

$$\equiv \frac{1}{\cos A \sin A}$$

$$\equiv \frac{1}{\frac{1}{2}\sin 2A}$$

$$\equiv \frac{2}{\sin 2A}$$

2) (a) $\frac{\cos 2A}{\sin^2 A} \equiv \operatorname{cosec}^2 A - 2$

Start with the left side

$$\frac{\cos 2A}{\sin^2 A} \equiv \frac{1 - 2\sin^2 A}{\sin^2 A}$$

$$\equiv \frac{1}{\sin^2 A} - 2$$

$$\equiv \operatorname{cosec}^2 A - 2$$

(b) $\frac{\cos 2A}{\cos^2 A} \equiv 2 - \sec^2 A$

Start with the left side

$$\frac{\cos 2A}{\cos^2 A} \equiv \frac{2\cos^2 A - 1}{\cos^2 A}$$

$$\equiv 2 - \frac{1}{\cos^2 A}$$

$$\equiv 2 - \sec^2 A$$

(c) $\frac{\cos 2A}{\sin^2 A} + \frac{\cos 2A}{\cos^2 A} \equiv \cot^2 A - \tan^2 A$

Use answers from (a) and (b) otherwise the solution may be penalised.

Worksheet G: Answers continued

$$\frac{\cos 2A}{\sin^2 A} + \frac{\cos 2A}{\cos^2 A} \equiv \operatorname{cosec}^2 A - 2 + 2 - \sec^2 A$$

$$\equiv \operatorname{cosec}^2 A - \sec^2 A$$

$$\equiv \cot^2 A + 1 - (1 + \tan^2 A)$$

$$\equiv \cot^2 A - \tan^2 A$$

3) $4\operatorname{cosec}^2 x \equiv \frac{\sec x}{\cos x} + \frac{\operatorname{cosec} x}{\sin x}$

Start with the right side

$$\frac{\sec x}{\cos x} + \frac{\operatorname{cosec} x}{\sin x} \equiv \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}$$

$$\equiv \frac{\sin^2 x + \cos^2 x}{\cos^2 x \sin^2 x}$$

$$\equiv \frac{1}{\cos^2 x \sin^2 x}$$

Note each step of the proof should be shown throughout.

$$\equiv \frac{1}{(\cos x \sin x)^2}$$

$$\equiv \frac{1}{\left(\frac{\sin 2x}{2}\right)^2}$$

$$\equiv \frac{1}{\frac{\sin^2 2x}{4}}$$

$$\equiv \frac{4}{\sin^2 2x}$$

$$\equiv 4\operatorname{cosec}^2 2x$$

4) $\cos 3x \equiv 4\cos^3 x - 3\cos x$

Start with the left side

$$\cos 3x \equiv \cos(2x + x)$$

$$\equiv \cos 2x \cos x - \sin 2x \sin x$$

$$\equiv (2\cos^2 x - 1)\cos x - 2\sin x \cos x \sin x$$

Worksheet G: Answers continued

$\equiv 2\cos^3 x - \cos x - 2\sin^2 x \cos x$

$\equiv 2\cos^3 x - \cos x - 2\,(1 - \cos^2 x)\cos x$

$\equiv 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$

$\equiv 4\cos^3 x - 3\cos x$

Worksheet H: Answers

$\tan 2x \cot x \equiv \frac{\sin 2x}{\cos 2x} \cot x \equiv \frac{2\cos x \cos x}{\cos 2x} \equiv \frac{2\cos^2 x}{\cos 2x}$ (this is given in the question)

$\cot x$ should firstly be written as $\frac{\cot x}{\sin x}$

$\sin 2x$ should be firstly written as $2\sin x \cos x$ before cancelling $\sin x$ in numerator and denominator

A better solution would be

$\tan 2x \cot x$

$\equiv \frac{\sin 2x \cos x}{\cos 2x \sin x}$

$\equiv \frac{2\sin x \cos x \cos x}{\cos 2x \sin x}$

$\equiv \frac{2\cos^2 x}{\cos 2x}$

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