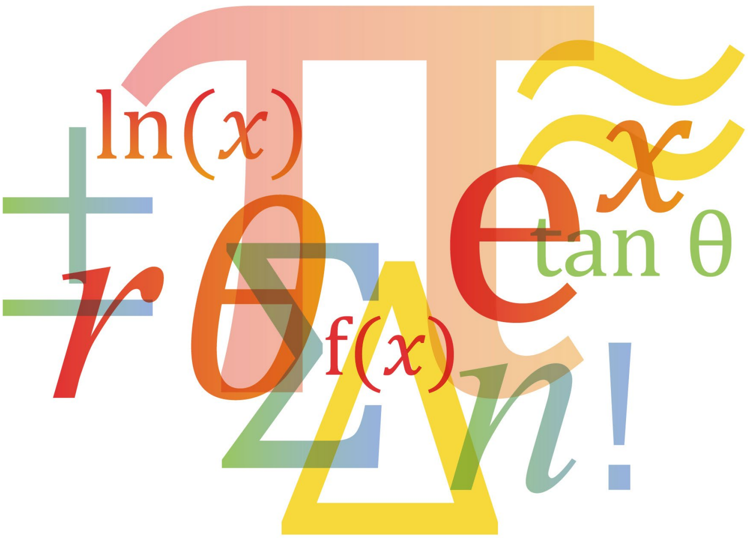
Cambridge Assessment
International Education

Teaching Pack

6.1 The Poisson Distribution

Cambridge International AS & A Level
Mathematics 9709



Version 1

Cambridge Pathway

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Introduction

This pack will help you to develop your learners' skills in mathematical thinking and mathematical communication, which are essential for success at AS & A Level and in further education.

Mathematical thinking and communication will be developed by focusing on:

1. Conceptual understanding – the 'why' behind the 'what'
2. Strategic competence – forming and solving problems
3. Adaptive reasoning – explanations, justifications and deductive reasoning

Throughout all activities, the learners will also develop:

- procedural fluency – know when, how and which rules to use
- positive disposition – believe maths can be learned, applied and is useful
- their skills in writing mathematically – writing working & proofs

These link to the course Assessment Objectives (AOs) which you can find in detail in the syllabus:

A01 Knowledge and understanding

A02 Application and communication

Each *Teaching Pack* contains one or more lesson plans and associated resources, complete with a section of preparation and reflection.

Each lesson is designed to be an hour long but you should adjust the timings to suit the lesson length available to you and the needs of your learners.

Important note
Our *Teaching Packs* have been written by **classroom teachers** to help you deliver topics and skills that can be challenging. Use these materials to supplement your teaching and engage your learners. You can also use them to help you create lesson plans for other topics.

This content is designed to give you and your learners the chance to explore a more active way of engaging with mathematics that encourages independent thinking and a deeper conceptual understanding. It is not intended as specific practice for the examination papers.

The *Teaching Packs* are designed to provide you with some example lessons of how you might deliver content. You should adapt them as appropriate for your learners and your centre. A single pack will only contain at most five lessons, it will **not** cover a whole topic. You should use the lesson plans and advice provided in this pack to help you plan the remaining lessons of the topic yourself.

Lesson preparation

This *Teaching Pack* will cover the following syllabus content:

Candidate should be able to:	Notes and examples
• use formulae to calculate probabilities for the distribution $\mathrm{Po}(\lambda)$	
• use the fact that if $X \sim \mathrm{Po}(\lambda)$ then the mean and variance of X are each equal to λ	Proofs are not required.
• understand the relevance of the Poisson distribution to the distribution of random events, and use the Poisson distribution as a model	
• use the Poisson distribution as an approximation to the binomial distribution where appropriate	The condition that n is large and p is small should be known; $n > 50$ and $np < 5$, approximately.

The remaining bullet point for topic 6.1 The Poisson distribution is not covered in this *Teaching Pack* (see the syllabus for detail). You will also need to include a lesson that covers a mixture of all the bullet points in exam-style questions. You will need to write your own lesson plans for these items.

Candidate should be able to:	Notes and examples
• use the normal distribution, with continuity correction, as an approximation to the Poisson distribution where appropriate	The condition that λ is large ahould be known; $\lambda > 15$, approximately.

Prior knowledge and skills

For all lessons, it is assumed that learners have already completed Cambridge IGCSE™ Mathematics 0580, or a course at an equivalent level. See the syllabus for more details of the expected prior knowledge for taking Cambridge International AS & A Level Mathematics 9709.

When planning any lesson, make a habit of always asking yourself the following questions about your learners' prior knowledge and skills:

- Do I need to re-teach this or do learners just need some practice?
- Is there an interesting activity that will efficiently achieve this?

Learners must have a good understanding of the mean and variance. It may be worth giving learners some simple data sets to use to calculate the mean and variance as practice before starting this topic:

Key learning objectives

The following list represents the main underlying concepts that you should make sure your learners have understood by the end of this topic:

- A Poisson distribution is used to model the number of random events occurring in a specified time or space.
- Probabilities of Poisson events occurring can be calculated given a constant rate.

- The Poisson distribution can be used as an approximation for the binomial distribution when *n* is large and *p* is small.
- The normal distribution can be used as an approximation for the Poisson distribution when $\lambda > 15$.

Why this topic matters

As well as being assessed as a topic in its own right, the skills in this pack are also used when solving problem involving linear combinations and hypothesis testing including calculating probabilities of type I and type II errors.

Key terminology and notation

Your learners will need to be confident with the following terminology and notation.

Binomial distribution $\boldsymbol{X \sim B(n, p)}$ – a frequency distribution of the possible number of successful outcomes in a given number of trials, n, where the probability of success, p, is constant.

Mean – an average value obtained by dividing the sum of the values by the number of values.

Parameter – a numerical quantity that characterises a given population.

Poisson distribution $\boldsymbol{X \sim Po(\lambda)}$ – a discrete frequency distribution that gives the probability of a given number of independent events occurring in a fixed period of time or space.

Variance – the average of the squared differences from the mean.

Insights video

There is an Insights video linked to this *Teaching Pack*:

- **6.1 The Poisson Distribution** – use this video before teaching Lesson 1, which will look at some of the misunderstanding learners have around the Poisson distribution, and whether or not they can use the Poisson distribution as a method to predict outcomes in a situation, or particular events happening over a period of time.

Teacher tutorials

There are *three* tutorials linked to this *Teaching Pack*:

- **Features of a Poisson distribution** – Review this tutorial before teaching Lesson plan 1; this will introduce the conditions needed to use the Poisson distribution.
- **Calculating probabilities using the Poisson distribution** – Review this tutorial before teaching Lesson plan 2; this will show you how to use the Poisson distribution to calculate probabilities and deal with changes in time/space period.
- **Using the Poisson distribution as an approximation to the binomial distribution** – Review this tutorial before teaching Lesson plan 3. This will show you how to demonstrate when the approximation can be used.

Lesson progression

Lesson 1 covers the second and third bullet points of syllabus content. Lesson 2 focuses on the first bullet point then leads to exam-style questions incorporating the second and third bullet points into exam-style questions. Once learners understand how to calculate probabilities using the Poisson distribution Lesson 3 introduces using the Poisson distribution as an approximation for the binomial distribution.

Going forward

This topic links to 6.2 Linear combinations of random variables and 6.5 Hypothesis tests.

Lesson plan 1: Modelling with the Poisson distribution

Preparation	• Review the Teacher tutorial *Features of a Poisson distribution*. • Cut paper into 8 cm x 8 cm squares if required.

Resources	• Paper, Mini whiteboards or other writing materials, rulers • Lesson slides *Features of a Poisson distribution* • Worksheet A: *Modelling using the Poisson distribution* • Worksheet B: *Further modelling using the Poisson distribution*

Learning objectives	By the end of the lesson: • ***all*** learners should know the conditions required to use the Poisson distribution as a model • **most** learners should be able to explain the conditions required to use the Poisson distribution in context • **some** learners should be able to come up with their own Poisson model

Dependencies

Learners need to know how to find the mean and variance of a given set of data (syllabus reference 5.1).

Common misconceptions

Misconception	Problems this can cause	An example way to resolve the misconception
If the mean and variance are equal then the data must follow a Poisson distribution.	Learners may only test this condition and not consider the other requirement of a Poisson distribution.	Show learners a set of data with an equal mean and variance that does not follow a Poisson distribution, packages such as Autograph or Desmos are best for this.

Timings	Activity
10 min	**Starter/Introduction** **Investigation** Lesson slides *Modelling with the Poisson Distribution* (slides 2-3) Learners need to understand that the Poisson distribution is related to random events. This starter introduces the idea of what is random. 1) Give learners a plain piece of paper and ask them to draw a square 8 cm x 8 cm (you could pre-cut paper to size to save time). 2) Ask them to put 16 crosses in the square at **random.** 3) They then split the square into 16 smaller 2 cm x 2 cm squares. 4) They count the number of crosses in each square then calculate the mean and variance of the 16 numbers. Explain to learners that if the points are truly random they should follow a Poisson distribution and the mean and variance should both be 1. The closer the variance is to 1, the more random the learners have been.

Timings	Activity
	Learners often expect random to mean that one cross ends up in each square but it's important to emphasise that you would get clusters of points as well as open spaces. (Note: you would expect approximately 6 squares with 0 crosses, six squares with 1 cross, three squares with 2 crosses and 1 square with 3 crosses.)
10 min	**Main lesson** Lesson slides *Modelling with the Poisson Distribution* (slides 4-6) Explain to learners that the standard conditions for use of the Poisson distribution for modelling purposes are that the events: • occur randomly (they are independent of each other) • occur singly (no chance of two events occurring at precisely the same point in time/space) • occur at a constant rate over time/space • data check – mean and variance approximately equal. Ask learners: Does the number of cars passing a point on a road follow a Poisson distribution? Allow learners to think-pair-share before taking feedback from pairs. Example responses: • it depends on the volume of traffic/time of day • the road must not have traffic lights or roadworks • it depends on the time period used. Explain to learners that this could be modelled using a Poisson distribution for a short period of time provided the road has no traffic lights, no roadworks and traffic is free-flowing (no traffic jams).
20 min	Worksheet A: *Modelling using the Poisson distribution* Give learners Worksheet A and ask them to work in pairs to decide whether the situations could be modelled using a Poisson distribution. They should give reasons for all the situations they do and do not think can be modelled using the Poisson distribution. Ask each pair to share their answer to a different question to the group and make sure they explain their reasons in context. Allow another pair to discuss whether they agree or disagree.
5 min	Lesson slides *Modelling with the Poisson Distribution* (slides 7–9) Refer back to the starter activity and explain that when using a Poisson distribution, the mean and variance should be equal. Work through the example in the PowerPoint presentation and emphasise that answers should be given in context.
10 min	Worksheet B: *Further modelling using the Poisson distribution* Learners complete Worksheet B. Check that answers are always given in context. **Extension:** Alternatively, if you have a suitable road nearby, get learners to record how many cars (or vans, bikes etc.) pass a point each minute. Repeat for 20 mins and calculate the mean and variance to test whether a Poisson distribution is suitable. **Challenge:** Ask learners to come up with a model where the mean and variance is the same but it does not follow a Poisson distribution – use Making Statistics Vital. **Support:** Some learners may need support calculating the mean and variance.

Timings	Activity
5 min	**Plenary** **Exit ticket** In their book, ask learners to write down three points they have learned about the Poisson distribution in today’s lesson. Ask learners to give one example of a situation that can be modelled by the Poisson distribution and share it with the class.

Reflection	Reflect on your lesson; use the **Lesson reflection** notes to help you.

Lesson plan 2: Calculating probabilities using the Poisson distribution

Preparation	• Cut up the cards for jigsaw activity Worksheet D: *Calculating probabilities jigsaw* (1 set between 2). • Review the Teacher tutorial *Calculating probabilities using the Poisson distribution.*

Resources	• Paper, Mini whiteboards or other writing materials • Worksheet C: *Inequality matching* • Worksheet D: *Calculating probabilities jigsaw* • Worksheet E: *Calculating probabilities exam style* • Worksheet F: *Spot the mistakes* • *Calculating probabilities using the Poisson distribution* PowerPoint

Learning objectives	By the end of the lesson: • ***all*** learners should be able to calculate probabilities using the formula when given a Poisson distribution • **most** learners should be able to calculate probabilities in exam-style written questions • **some** learners should be able to find values for X or λ when given a probability

Dependencies

Syllabus section 5.3 Probability

Common misconceptions

Misconception	Problems this can cause	An example way to resolve the misconception
Turning probability statements into correct inequalities. For example 'at least 3', 'no more than 3' or 'at most 3' do not include 3 and 'more than 3' or 'less than 3' do include 3.	Learners often confuse $P(X < 3)$ with $P(X \leq 3)$ or $P(X > 3)$ with $P(X \geq 3)$.	The starter activity is designed to identify problems and encourage learners to use number lines to find the correct values to include.
That the average rate in time/space is always the same.	Although the rate is constant it must be adjusted for the correct time/space period.	Use the second example to highlight changes in time period. All questions in Worksheet E include a change in time/space.

Timings	Activity
5 min	**Starter/Introduction** Worksheet C: *Inequality matching* Learners match up the statements to the appropriate inequalities. This will help them to turn worded questions into the inequalities they require. Get learners to come up to the board to match up pairs after completing the sheet. Make sure that all learners are happy with the vocabulary as they will need to turn worded exam-style questions into inequalities later in the lesson. **Support:** Encourage less able learners to draw number lines.
5 min	**Main lesson** Introduce the notation $X \sim Po(\lambda)$ where $\mu = \sigma^2 = \lambda$ and $p_r = e^{-\lambda}\frac{\lambda^r}{r!}$ as given in the formula book. Lesson slides *Calculating probabilities using the Poisson distribution* (slides 4–5) It is important that learners are confident calculating probabilities using the formula without any context before moving onto exam-style questions. Complete the examples on the PowerPoint presentation. Encourage learners to take out the exponential factor when calculating multiple probabilities and to give answers to more than 3 significant figures (sf) before rounding.
20 min	*Worksheet D: Calculating probabilities jigsaw* Give learners the jigsaw from Worksheet D and get them to complete it in pairs. They may find additional paper or mini whiteboards useful to complete working out. As learners are completing the jigsaw move around the pairs and help with any problem cards, this is a good opportunity to target less able learners. Many questions have similar answers so they must check rounding carefully, and two questions have the same answer so they must match carefully. **Challenge:** Given that $P(X = 1) = 0.0630$ find the value of λ to 2sf. (Answer $\lambda = 4.2$) **Support:** Give learners who need more support part of the jigsaw or some pairs already completed. Encourage them to draw number lines.
5 min	Lesson slides *Calculating probabilities using the Poisson distribution* (slides 7–10) Learners should now all be able to find probabilities using the formula. This needs to be extended to exam-style questions in context. Complete the examples from the PowerPoint presentation. It is important to get learners to state which distribution they are using and emphasise that the average rate needs to be adjusted for the given period of time or space.
15 min	Worksheet E: *Calculating probabilities exam style* Learners complete Worksheet E. As learners are completing the questions it is important to check that they are using correct notation and the correct distribution.

Timings	Activity
	Challenge: Extension question on Worksheet E then learners should create their own exam- style question and mark scheme. **Support**: Learners may need support adjusting the average rate. Where they struggle with the inequalities get them to draw number lines to select the appropriate values.
10 min	**Plenary** Lesson slides *Calculating probabilities using the Poisson distribution* (slides 12–13) Worksheet F: *Spot the mistakes* Worksheet F includes an exam question with a completed solution containing common mistakes. Learners should act like the teacher and try to find and correct the mistakes. Ask learners to feed back the mistakes they have found and ask them to model the corrections on the board.

Reflection	Reflect on your lesson; use the **Lesson reflection** notes to help you.
How did the pairings for the jigsaw activity work?	Did you pair learners of a similar ability or mixed ability? Did all learners contribute to the work equally?

Lesson plan 3: The Poisson distribution as an approximation to the binomial distribution

Preparation	• Review the Teacher tutorial *The Poisson distribution as an approximation to the binomial distribution.*

Resources	• Paper, Mini whiteboards or other writing materials • Worksheet G: *Calculating probabilities* • Worksheet H: *Dice investigation* • Dice – two per learner • Worksheet I: *Exam-style questions* • Lesson slides *The Poisson distribution as an approximation to the binomial distribution*

Learning objectives	By the end of the lesson: • ***all*** learners should recognise when a Poisson distribution can be used as an approximation to the binomial distribution • **most** learners should know the approximation $X \sim Po(np)$ and be able to use it to calculate probabilities • **some** learners should be able to find n or p given a probability

Dependencies

Learners need to know how to find probabilities using the binomial distribution, syllabus section 5.4.

Common misconceptions

Misconception	Problems this can cause	An example way to resolve the misconception
That the binomial distribution can always be approximated using a Poisson distribution.	Learners may use an approximation when an exact probability should be calculated.	The starter and dice investigation should highlight when the distributions are approximately equal and emphasise the requirements of n large and p small.

Timings	Activity
10 min	**Starter/Introduction** Lesson slides The Poisson distribution as an approximation to the binomial distribution (slide 2) Worksheet G: *Calculating probabilities* Give each learner a sheet and ask them to calculate the probabilities for the given binomial and Poisson distribution questions. When they have calculated all the probabilities, ask them to compare the probabilities. Encourage learners to come up with their own probability distributions to test any patterns they have found. Have a class discussion about when they get similar answers. They should be able to spot that the probabilities become more similar as n gets larger and p gets smaller.

Timings	Activity
20 min	**Main lesson** Lesson slides *The Poisson distribution as an approximation to the binomial distribution* (slides 3-4) Worksheet H: *Dice investigation* Give each learner a pair of dice to roll and record the number of learners who get two sixes (success). Repeat this experiment 20 times to give 20 observations from a binomial distribution $X \sim B\left(n, \frac{1}{36}\right)$ where n is the class size. Learners record this information on Worksheet H. The number of rows in the table may need to be extended for larger class sizes but it is unlikely that in most classes you will get more than ten successes at once. Once the data is collected, learners calculate the relative frequencies, the exact binomial probabilities and the approximate Poisson probabilities using $X \sim Po(np)$. Get all learner to write down the distributions in the space provided using the appropriate n for class size before they start to calculate the probabilities. Encourage learners to compare the probabilities and discuss how good the approximation is. You could also use graphing software such as Autograph or Desmos to draw the distributions. As p is so small learners should find it is a good approximation. **Support:** Less able learners may need support calculating the probabilities.
10 min	Lesson slides *The Poisson distribution as an approximation to the binomial distribution* (slides 5-8) Explain to learners that to use the Poisson distribution as an approximation to the binomial we require n to be large and p to be small, approximately $n > 50$ and $np < 5$. We use $\lambda = np$. Complete the example on the PowerPoint presentation to model a question in context.
15 min	Worksheet I: *Exam-style questions* Give learners Worksheet I so they can complete exam-style questions using the approximation. While learners are completing the questions, circulate to make sure that learners are using the models and inequalities. **Challenge:** Extension question on Worksheet I.
5 min	**Plenary** Review the exam questions and get learners to self-assess their work. Each learner should fill in a small slip of paper or post-it note with a score from 1 (Very confident) to 3 (Not confident) and a question they'd like to be answered. Collect these to use for planning the next lesson.

Reflection	Reflect on your lesson; use the **Lesson reflection** notes to help you.
Use the learners' self-assessments to judge how well they feel they understand the topic. Do you need to do another lesson or are they ready to move on?	

Planning your own lessons

You now need to plan lessons to cover the following bullet points:

Candidate should be able to:	**Notes and examples**
• use the normal distribution, with continuity correction, as an approximation to the Poisson distribution where appropriate	The condition that λ is large ahould be known; $\lambda > 15$, approximately.

Follow the structure of the *Teaching Pack*, and use techniques from the 'How to' guides, to create your own engaging lessons to cover these bullet points. Consider what preparation you need for each lesson: what prior knowledge is needed, what are the key objectives, what are the dependencies, what common misconceptions are there, and so on.

Below, we have provided an outline of some activities and approaches you might like to try.

Lesson 4: Using the normal distribution as an approximation to the Poisson distribution
Common misconceptions: Learners don't have a clear understanding of how calculating probabilities differs between discrete and continuous distributions and so often fail to use continuity correction or use them incorrectly.
Starter: You could try some revision of finding probabilities using the normal distribution.
Main: You could have a class discussion about continuity correction and do lots of practice of these using number lines before introducing the approximation.
Plenary: You could try giving students a completed solution to a question cut up into steps for them to order. Include multiple versions of each step some of which are incorrect so students can identify the common errors, in particular multiple attempts of the continuity correct would be good so again reinforce the common errors.

You will find some other activity suggestions in the Scheme of Work.

Lesson reflection

As soon as possible after the lesson you need to think about how well it went.

One of the key questions you should always ask yourself is:

Did all learners get to the point where they can access the next lesson? If not, what will I do?

Reflection is important so that you can plan your next lesson appropriately. If any misconceptions arose or any underlying concepts were missed, you might want to use this information to inform any adjustments you should make to the next lesson.

It is also helpful to reflect on your lesson for the next time you teach the same topic. If the timing was wrong or the activities did not fully occupy the learners this time, you might want to change some parts of the lesson next time. There is no need to re-plan a successful lesson every year, but it is always good to learn from experience and to incorporate improvements next time.

To help you reflect on your lesson, answer the most relevant questions below.

Were the lesson objectives realistic?
What did the learners learn today? Or did they learn what was intended? Why not?
What proportion of the time did we spend on the most important topics?
Were there any common misconceptions?
What do I need to address next lesson?
What was the learning atmosphere like?
Did my planned differentiation work well?
How could I have helped the lowest achieving learners to do more?
How could I have stretched the highest achieving learners even more?
Did I stick to timings?
What changes did I make from my plan and why?

Summary evaluation

What two things went really well? (Consider both teaching and learning.)

What two things would have improved the lesson? (Consider both teaching and learning.)

What have I learned from this lesson about the class or individuals that will inform my next lesson?

Worksheets and answers

Worksheet A: Modelling using the Poisson distribution

For each of the situations below, decide whether or not they can be modelled by a Poisson distribution. Give at least one condition for each situation that **can** be modelled using a Poisson distribution and for those you think **cannot** be modelled using a Poisson distribution explain why.

Situation	Poisson distribution?
1. The number of phone calls received by a bank per day.	
2. Number of cars passing a point on a very busy motorway in a 10-minute period.	
3. The number of people waiting at a bus stop per hour each day.	
4. The number of particles emitted per minute by a radioactive substance.	
5. The number of accidents in a large factory per month.	
6. The number of injuries in a large factory per month.	
7. The number of errors per 10 pages of a first draft of a new book.	
8. The number of weeds growing in a randomly selected square metre of field.	
9. The number of patients in a hospital with an infectious disease per month.	
10. There are four different queues in a shop; the number of people joining each queue per hour.	

Worksheet B: Further modelling using the Poisson distribution

Investigate whether any of the situations below are suitable to be modelled using a Poisson distribution.

1) The number of coffees sold per hour in a coffee shop is recorded throughout the day:

Hour 1	**Hour 2**	**Hour 3**	**Hour 4**	**Hour 5**	**Hour 6**	**Hour 7**	**Hour 8**	**Hour 9**	**Hour 10**
45	67	52	22	12	72	54	34	12	20

Give one reason based on the data and one other reason why a Poisson distribution is unlikely to be a suitable model.

2) The number of TVs sold per day in a large electrical store over a 10-day period is recorded below:

Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Day 6**	**Day 7**	**Day 8**	**Day 9**	**Day 10**
5	9	5	12	3	19	11	3	15	9

Comment on the suitability of a Poisson distribution to model this situation.

Worksheet B: Further modelling using the Poisson distribution continued

3) On average, a stretch of road has 16 accidents per month. Give two assumptions needed to model this as a Poisson distribution and a limitation of this model.

4) A taxi company's records show the number of breakdowns that occur each week:

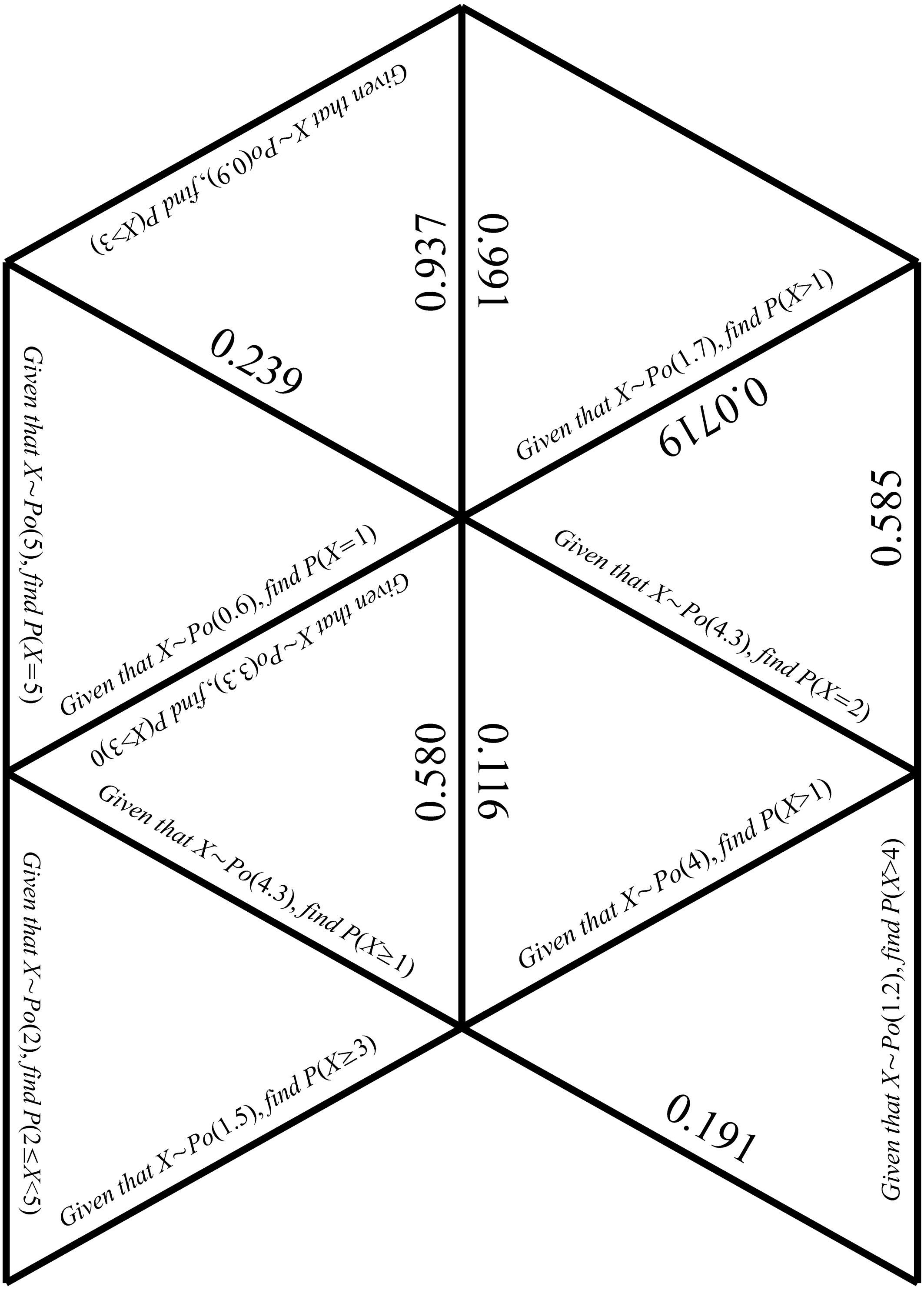
Number of breakdowns per week	0	1	2	3	4
Frequency	11	8	12	10	10

Comment on the suitability of a Poisson distribution to model this situation.

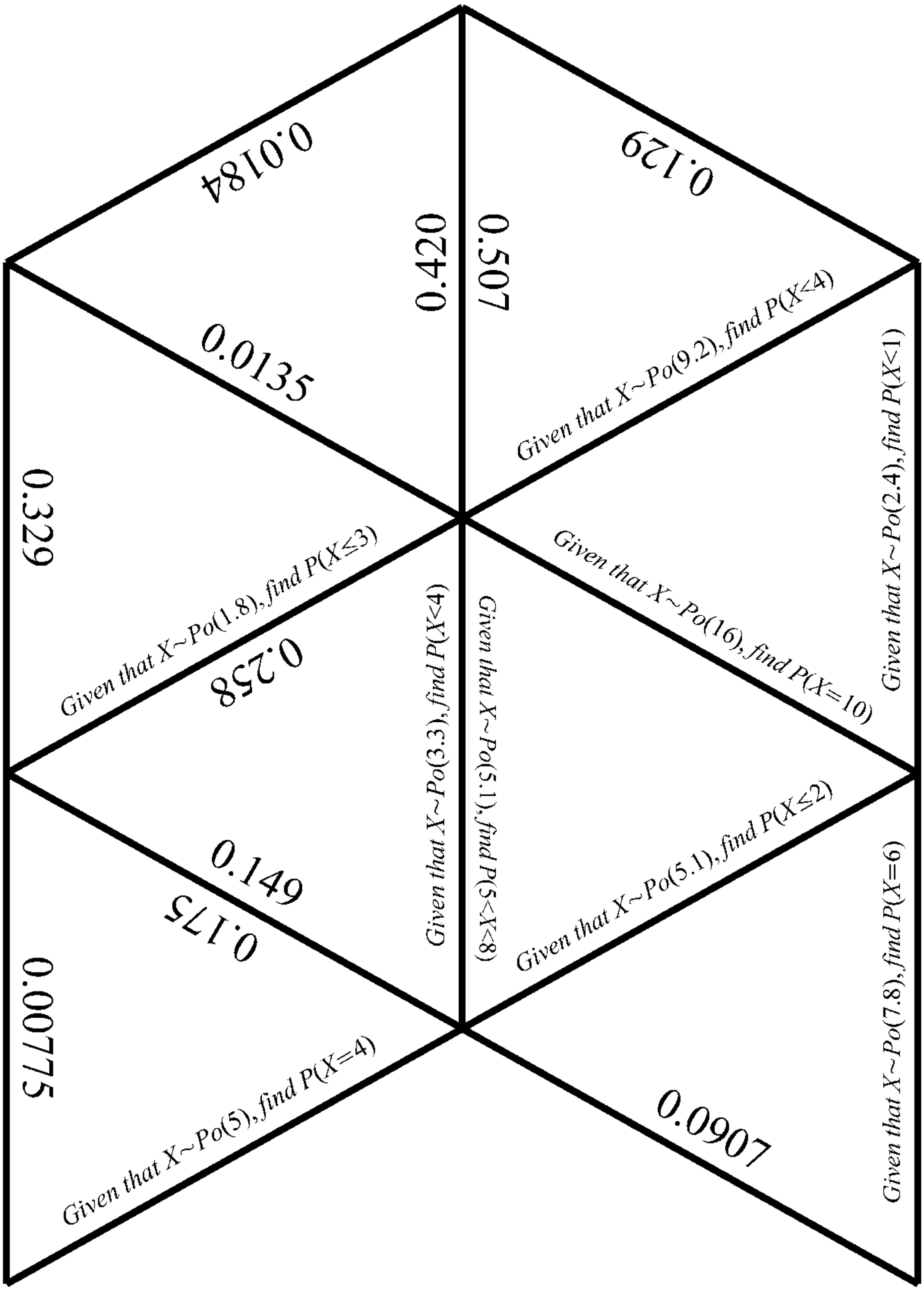
Worksheet C: Inequality matching

At most five	$P(x \leq 9) - P(x \leq 3)$
More than five but fewer than nine	$P(x \leq 2)$
Exactly six	$P(x \leq 8) - P(x \leq 5)$
At least three	$1 - P(x \leq 9)$
At least four but fewer than ten	$P(x = 6)$
Seven or more	$1 - P(x \leq 6)$
Between three and six inclusive	$P(x \leq 7)$
No more than seven	$1 - P(x \leq 2)$
Less than three	$P(x \leq 5)$
More than nine	$P(x \leq 6) - P(x \leq 2)$

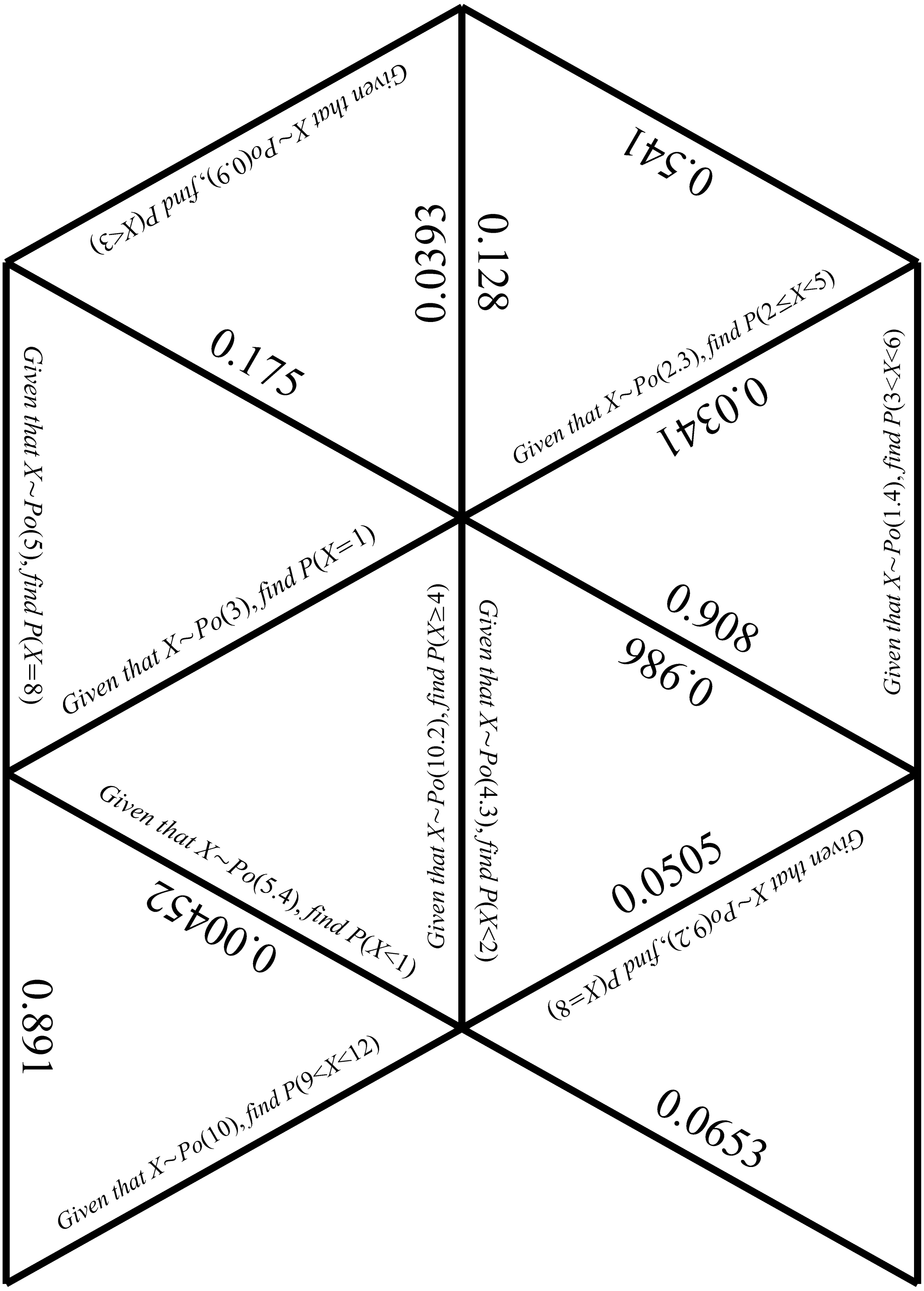
Worksheet D: Calculating probabilities jigsaw



Worksheet D: Calculating probabilities jigsaw continued



Worksheet D: Calculating probabilities jigsaw continued



Worksheet E: Calculating probabilities exam style

1) The number of customers arriving at a supermarket checkout can be modelled as a Poisson distribution with mean 2.3 per 10-minute interval. Find the probability that:
   a) No customers arrive between 0950 and 1000.
   b) 3 customers arrive between 1400 and 1420.
   c) 1 customer arrives between 1830 and 1835.

2) An insurance company helpline receives on average 3 calls every 5 minutes, assuming the calls follow a Poisson distribution. Find the probability that:
   a) they will receive fewer than 5 calls in a 10-minute period
   b) they will receive exactly 9 calls in a 30-minute period
   c) they will receive more than 3 but fewer than 6 calls in a 15 minute period.

Worksheet E: Calculating probabilities exam style continued

3) Faults in a certain piece of fabric occur at random and independently. On average the faults occur at a rate of 2 per metre. Find the probability that:
   a) more than 3 faults will occur in a piece of fabric 2 metres long
   b) between 2 and 5 faults inclusive will occur in a piece of fabric 150 cm long.

Extension question

Given that the probability of receiving 4 calls in 10 minutes is twice as likely as receiving 5 calls in 10 minutes, find the average number of calls received per hour.

Worksheet F: Spot the mistakes

9709/73 May/June 2014

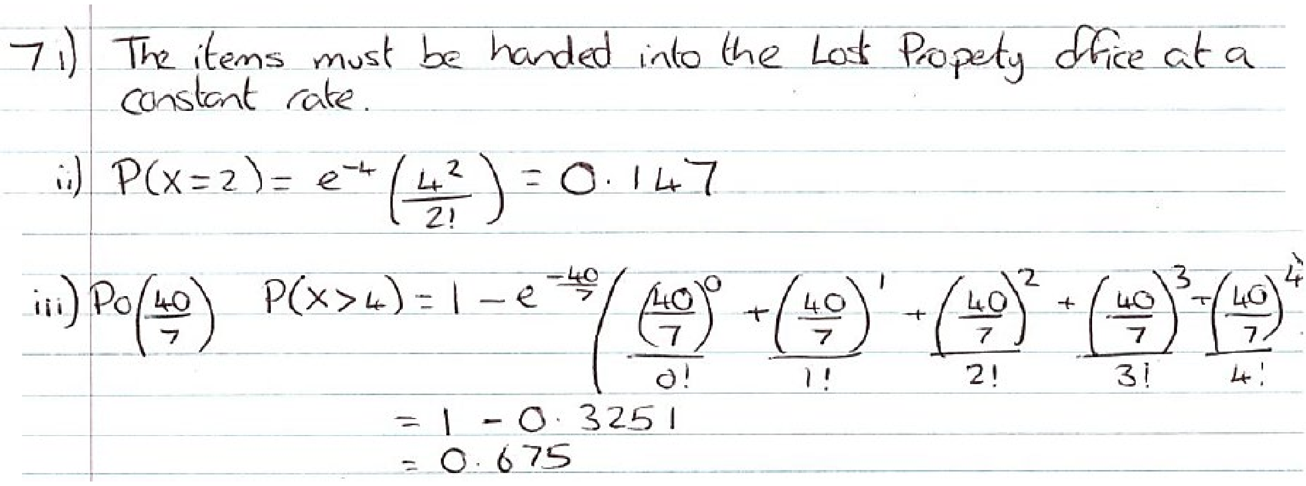
7 A Lost Property office is open 7 days a week. It may be assumed that items are handed in to the office randomly, singly and independently.

(i) State another condition for the number of items handed in to have a Poisson distribution. [1]

It is now given that the number of items handed in per week has the distribution Po(4.0).

(ii) Find the probability that exactly 2 items are handed in on a particular day. [2]

(iii) Find the probability that at least 4 items are handed in during a 10-day period. [3]



7 i) The items must be handed into the Lost Propety office at a constant rate.

ii) $P(x=2) = e^{-4}\left(\frac{4^2}{2!}\right) = 0.147$

iii) $Po\left(\frac{40}{7}\right)$ $P(x>4) = 1 - e^{-\frac{40}{7}}\left(\frac{\left(\frac{40}{7}\right)^0}{0!} + \frac{\left(\frac{40}{7}\right)^1}{1!} + \frac{\left(\frac{40}{7}\right)^2}{2!} + \frac{\left(\frac{40}{7}\right)^3}{3!} + \frac{\left(\frac{40}{7}\right)^4}{4!}\right.$

$= 1 - 0.3251$

$= 0.675$

Worksheet G: Calculating probabilities

Investigation – When do the binomial and Poisson distributions give similar answers?

Task 1

Binomial	Poisson
$X \sim B(12, 0.6)$ $P(X = 4)$	$X \sim Po(7.2)$ $P(X = 4)$
$X \sim B(60, 0.6)$ $P(X = 35)$	$X \sim Po(36)$ $P(X = 35)$
$X \sim B(50, 0.2)$ $P(X = 10)$	$X \sim Po(10)$ $P(X = 10)$
$X \sim B(50, 0.05)$ $P(X = 5)$	$X \sim Po(2.5)$ $P(X = 5)$
$X \sim B(100, 0.02)$ $P(X = 5)$	$X \sim Po(2)$ $P(X = 5)$

Task 2

Compare your probabilities, what patterns do you notice? Use additional examples to investigate your conclusion

Worksheet H: Dice investigation

- Each member of the class needs a pair of dice to roll.
- A success will be to roll two sixes,
- The total number of successes in the class should be recorded each time
- Repeat the experiment 20 times.

Then calculate the relative frequencies, the exact binomial probabilities using $X \sim B\left(n, \frac{1}{36}\right)$ and the approximate Poisson probabilities using $X \sim Po\left(\frac{1}{36} \times n\right)$.

Data collection
Number of learners ………………… so use the following:

Binomial distribution

Poisson distribution

X	Frequency, f	Relative frequency, $\frac{f}{20}$	Binomial probability	Poisson probability
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
……….				
	Total = 20			

Conclusion

Worksheet I: Exam-style questions

1) A random variable X has the distribution $X \sim B(1200, 0.09)$. Use the Poisson approximation to the binomial distribution to find $P(X > 2)$.

2) A particular disease is known to affect 0.3% of the population. A random sample of 300 people is taken.

   a) Write down the distribution of the data.
   b) Explain why the Poisson distribution would be a suitable approximation.
   c) Use the approximation to find the probability that fewer than four people in the sample have the disease.

3) A machine produces lightbulbs and it is know that 99% of the bulbs produced are fault-free. A random sample of 300 bulbs is tested for faults. X is the number of faulty bulbs in the sample.

   a) Write down the distribution of X and a suitable approximate distribution; give a justification for your approximation.
   b) Use the approximation to find $(X \geq 1)$.

Extension question
The random variable X follows the distribution $X \sim B(250, p)$.

Given that a Poisson approximation is used and $P(X = 3) = 0.15169$ and $P(X = 5) = 0.17475$ find the value of p to 2sf.

Worksheet A: Answers

Situation	Poisson Distribution?
1. The number of phone calls received by a bank per day.	Yes, reasonable to assume that the phone calls occur at random, independently, singly and the rate of calls is constant per day.
2. Number of cars passing a point on a very busy motorway in a 10-minute period.	No, as the motorway is busy the traffic will not flow freely so cars are not independent of each other.
3. The number of people waiting at a bus stop per hour each day.	No, the number of people waiting at a bus stop is unlikely to have a constant rate. More people will be getting buses at 9am than at 3am.
4. The number of particles emitted per minute by a radioactive substance.	Yes reasonable to assume that particles emitted occur at random, independently, singly and the rate of particles emitted is constant per minute.
5. The number of accidents in a large factory per month.	Yes reasonable to assume that the accidents occur at random, independently, singly and the rate of accidents is constant per month.
6. The number of injuries in a large factory per month.	No, one accident may cause multiple injuries so one injury may not be independent from another injury.
7. The number of errors per 10 pages of a first draft of a new book.	Yes reasonable to assume that the errors occur at random, independently, singly and the rate of errors is constant per 10 pages.
8. The number of weeds growing in a randomly selected square metre of field.	Yes reasonable to assume that weeds occur at random, independently, singly and the rate of weeds growing is constant per m^2.
9. The number of patients in a hospital with an infectious disease per month.	No, if the disease is infectious the event of getting the disease is not independent of others getting the disease.
10. There are four different queues in a shop; the number of people joining each queue per hour.	No, people are likely to join the shortest queue so the events are not random.

Worksheet B: Answers

Investigate whether any of the situations below are suitable to be modelled using a Poisson distribution.

1) The number of coffees sold per hour in a coffee shop is recorded throughout the day:

Hour 1	**Hour 2**	**Hour 3**	**Hour 4**	**Hour 5**	**Hour 6**	**Hour 7**	**Hour 8**	**Hour 9**	**Hour 10**
45	67	52	22	12	72	54	34	12	20

Give one reason based on the data and one other reason why a Poisson distribution is unlikely to be a suitable model.

Mean=39 Variance=443.6
The number of coffees sold is unlikely to be a constant rate throughout the day as more people buy coffee in the morning and during work breaks. The mean and variance are not equal so the Poisson distribution is unlikely to be a suitable model.

2) The number of TVs sold per day in a large electrical store over a 10-day period is recorded below:

Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Day 6**	**Day 7**	**Day 8**	**Day 9**	**Day 10**
5	9	5	12	3	19	11	3	15	9

Comment on the suitability of a Poisson distribution to model this situation.

Mean=9.1 Variance=25.29
The number of TVs sold is unlikely to be a constant rate throughout the week as more people will go shopping at the weekend. The mean and variance are not equal so the Poisson distribution is unlikely to be a suitable model.

3) On average, a stretch of road has 16 accidents per month. Give two assumptions needed to model this as a Poisson distribution and a limitation of this model.

This assumes that accidents happen at a constant rate each month and independently of other accidents, but there may be more accidents when the traffic is busy or the weather is bad. There may be more accidents in the winter when it's icy compare to summer months.

4) A taxi company's records show the number of breakdowns that occur each week:

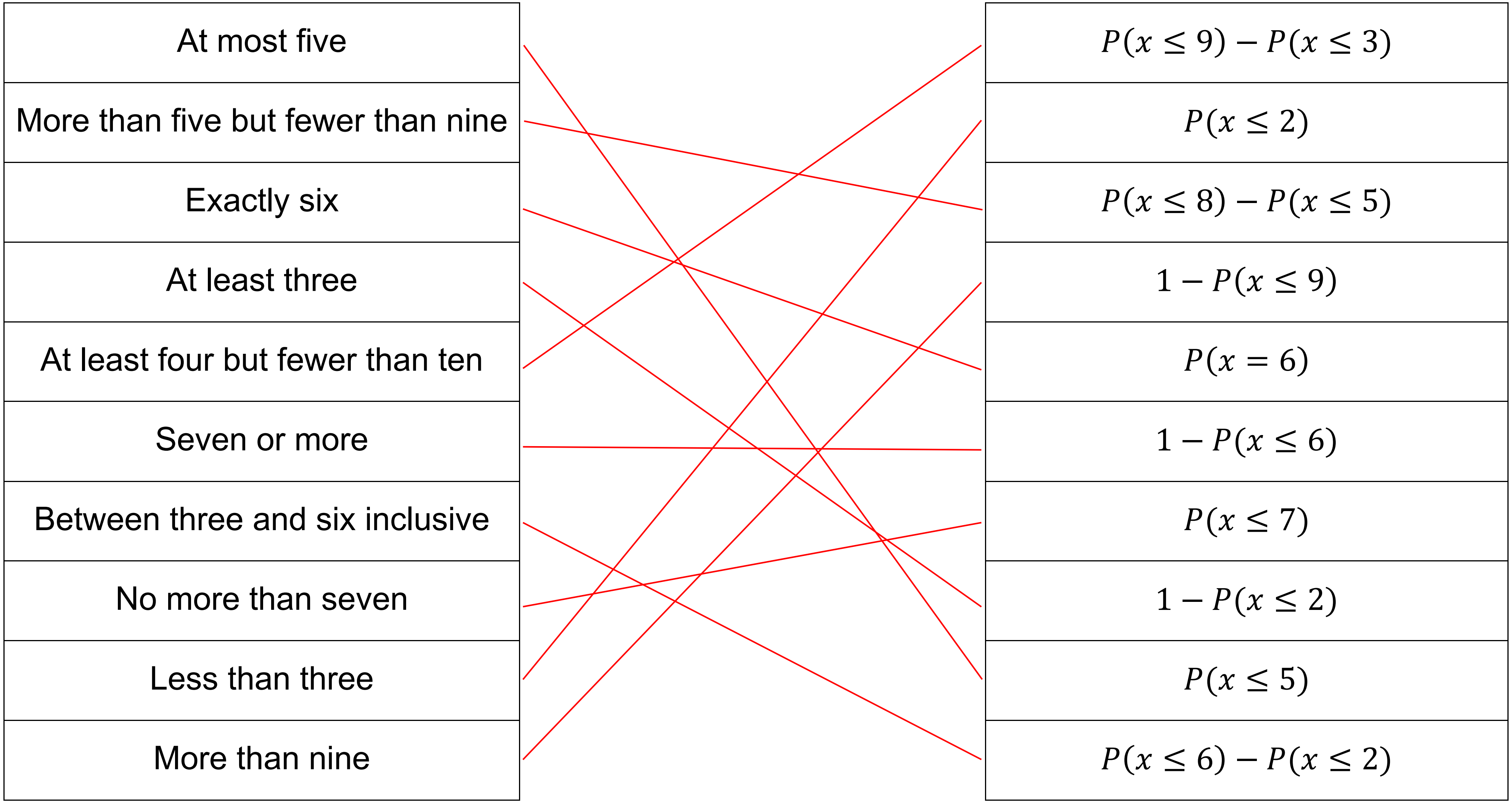
Number of breakdowns per week	0	1	2	3	4
Frequency	11	8	12	10	10

Comment on the suitability of a Poisson distribution to model this situation.

Mean=2 Variance=2

It is reasonable to assume that the taxis break down independently of each other, not at exactly the same time and at a constant rate. The mean and variance are equal so the Poisson distribution is likely to be a suitable model.

Worksheet C: Answers



At most five

More than five but fewer than nine

Exactly six

At least three

At least four but fewer than ten

Seven or more

Between three and six inclusive

No more than seven

Less than three

More than nine

$P(x \leq 9) - P(x \leq 3)$

$P(x \leq 2)$

$P(x \leq 8) - P(x \leq 5)$

$1 - P(x \leq 9)$

$P(x = 6)$

$1 - P(x \leq 6)$

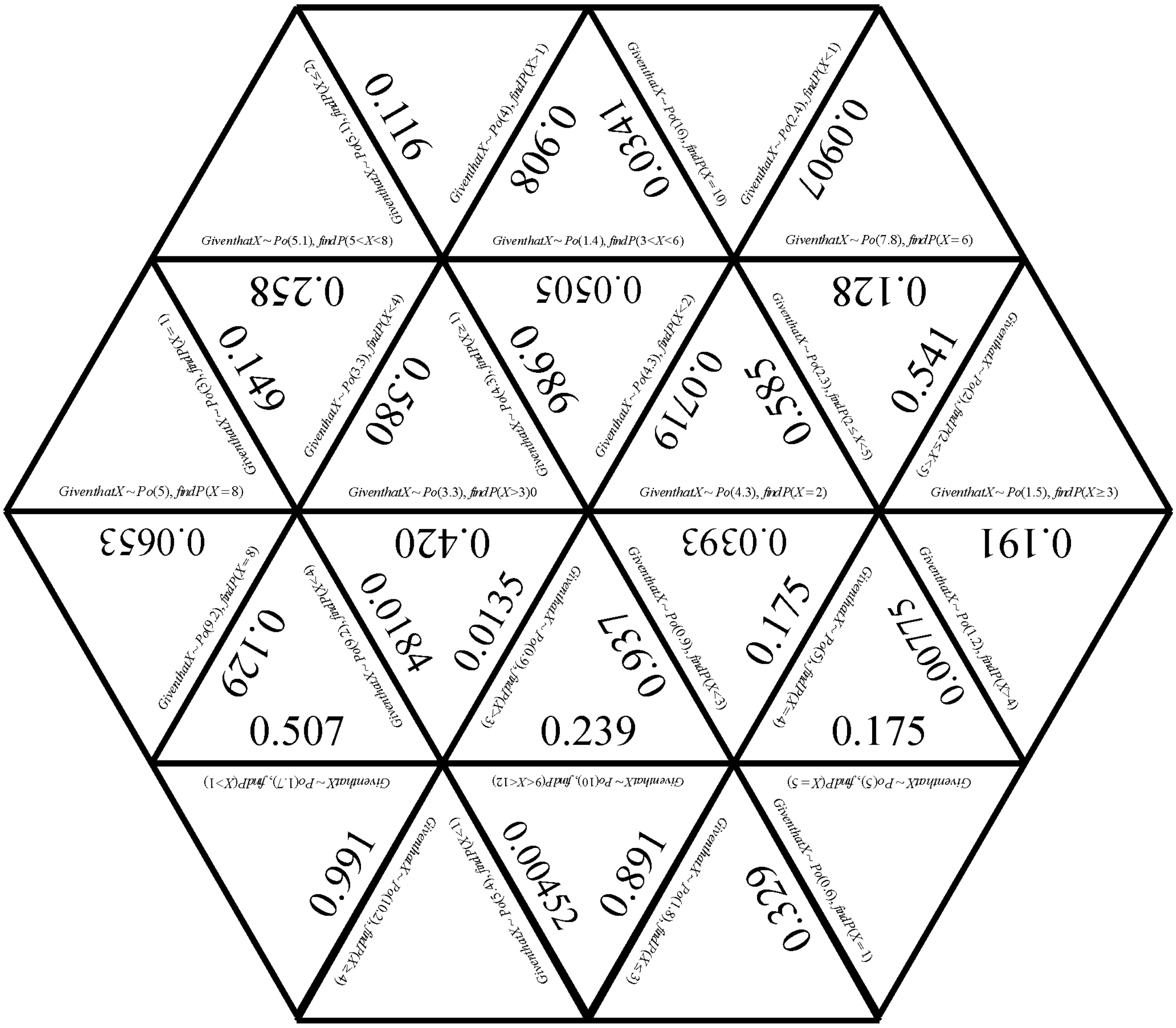
$P(x \leq 7)$

$1 - P(x \leq 2)$

$P(x \leq 5)$

$P(x \leq 6) - P(x \leq 2)$

Worksheet D: Answers



Worksheet E: Answers

1) The number of customers arriving at a supermarket checkout can be modelled as a Poisson distribution with mean 2.3 per 10-minute interval. Find the probability that:
   a) No customers arrive between 0950 and 1000

   $$X \sim Po(2.3)\ P(X=0) = e^{-2.3}\left(\frac{2.3^0}{0!}\right) = 0.100\ (3sf)$$

   b) 3 customers arrive between 1400 and 1420

   $$X \sim Po(4.6)\ P(X=3) = e^{-4.6}\left(\frac{4.6^3}{3!}\right) = 0.163\ (3sf)$$

   c) 1 customer arrives between 1830 and 1835

   $$X \sim Po(1.15)\ P(X=1) = e^{-1.15}\left(\frac{1.15^1}{1!}\right) = 0.364\ (3sf)$$

2) An insurance company helpline receives on average 3 calls every 5 minutes, assuming the calls follow a Poisson distribution. Find the probability that:
   a) they will receive fewer than 5 calls in a 10-minute period

   $$X \sim Po(6)\ P(X<5) = e^{-6}\left(\frac{6^0}{0!}+\frac{6^1}{1!}+\frac{6^2}{2!}+\frac{6^3}{3!}+\frac{6^4}{4!}\right) = 0.285\ (3sf)$$

   b) they will receive exactly 9 calls in a 30-minute period

   $$X \sim Po(18)\ P(X=9) = e^{-18}\left(\frac{18^9}{9!}\right) = 0.00833\ (3sf)$$

   c) they will receive more than 3 but fewer than 6 calls in a 15-minute period

   $$X \sim Po(9)\ P(3<X<6) = e^{-9}\left(\frac{9^3}{3!}+\frac{9^4}{4!}+\frac{9^5}{5!}\right) = 0.109\ (3sf)$$

3) Faults in a certain piece of fabric occur at random and independently. On average the faults occur at a rate of 2 per metre. Find the probability that
   a) more than 3 faults will occur in a piece of fabric 2 metres long

   $$X \sim Po(4)\ P(X>3) = 1 - e^{-4}\left(\frac{4^0}{0!}+\frac{4^1}{1!}+\frac{4^2}{2!}+\frac{4^3}{3!}\right) = 0.567\ (3sf)$$

   b) Between 2 and 5 faults inclusive will occur in a piece of fabric 150 cm long

   $$X \sim Po(3)\ P(2 \le X \le 5) = e^{-3}\left(\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\frac{3^5}{5!}\right) = 0.717\ (3sf)$$

Extension question

Given that the probability of receiving 4 calls in 10 minutes is twice as likely as receiving 5 calls in 10 minutes, find the average number of calls received per hour.

$$P(X=4) = 2P(X=5)$$

$$e^{-\lambda}\left(\frac{\lambda^4}{4!}\right) = 2e^{-\lambda}\left(\frac{\lambda^5}{5!}\right)$$

$$\lambda^4 = \frac{2\lambda^5}{5}$$

$$\lambda = \frac{5}{2}$$

Worksheet F: Answers

9709/73 May/June 2014

7 A Lost Property office is open 7 days a week. It may be assumed that items are handed in to the office randomly, singly and independently.

(i) State another condition for the number of items handed in to have a Poisson distribution. [1]

It is now given that the number of items handed in per week has the distribution Po(4.0).

(ii) Find the probability that exactly 2 items are handed in on a particular day. [2]

(iii) Find the probability that at least 4 items are handed in during a 10-day period. [3]

7 i) The items must be handed into the Lost Propety office at a constant rate. ✓

This is the rate per week

ii) $P(x=2) = e^{\boxed{-4}}\left(\frac{4^2}{2!}\right) = 0.147$

At least 4 should be ≥4

iii) $Po\left(\frac{40}{7}\right)$ $P(x \boxed{>4}) = 1 - e^{-\frac{40}{7}}\left(\frac{\left(\frac{40}{7}\right)^0}{0!} + \frac{\left(\frac{40}{7}\right)^1}{1!} + \frac{\left(\frac{40}{7}\right)^2}{2!} + \frac{\left(\frac{40}{7}\right)^3}{3!} + \cancel{\frac{\left(\frac{40}{7}\right)^4}{4!}}\right)$

$= 1 - 0.3251$

$= 0.675$

Corrections

ii) $X \sim Po\left(\frac{4}{7}\right)$ $P(x=2) = e^{-\frac{4}{7}}\left(\frac{\left(\frac{4}{7}\right)^2}{2!}\right) = 0.0922$ (3sf)

iii) $X \sim Po\left(\frac{40}{7}\right)$ $P(x \geq 4) = 1 - P(x \leq 3) = 0.821$ (3sf)

Worksheet G: Answers

Binomial	Poisson
$X \sim B(12,0.6)$ $P(X=4) = \binom{12}{4}(0.6)^4(1-0.6)^8$ $P(X=4) = 0.0420$	$X \sim Po(7.2)$ $P(X=4) = e^{-7.2}\left(\frac{7.2^4}{4!}\right)$ $P(X=4) = 0.0836$
$X \sim B(60,0.6)$ $P(X=35) = \binom{60}{35}(0.6)^{35}(1-0.6)^{25}$ $P(X=35) = 0.100$	$X \sim Po(36)$ $P(X=35) = e^{-35}\left(\frac{35^9}{9!}\right)$ $P(X=35) = 0.0663$
$X \sim B(50,0.2)$ $P(X=10) = \binom{50}{10}(0.2)^{10}(1-0.2)^{40}$ $P(X=10) = 0.140$	$X \sim Po(10)$ $P(X=10) = e^{-10}\left(\frac{10^{10}}{10!}\right)$ $P(X=10) = 0.125$
$X \sim B(50,0.05)$ $P(X=5) = \binom{50}{5}(0.05)^5(1-0.05)^{45}$ $P(X=5) = 0.0658$	$X \sim Po(2.5)$ $P(X=5) = e^{-2.5}\left(\frac{2.5^5}{5!}\right)$ $P(X=5) = 0.0668$
$X \sim B(100,0.02)$ $P(X=5) = \binom{100}{5}(0.2)^5(1-0.2)^{95}$ $P(X=5) = 0.0353$	$X \sim Po(2)$ $P(X=5) = e^{-2}\left(\frac{2^5}{5!}\right)$ $P(X=5) = 0.0361$

Worksheet I: Answers

1) A random variable X has the distribution $X \sim B(120, 0.009)$. Use the Poisson approximation to the binomial distribution to find $P(X > 2)$.

   $X \sim Po(1.08)$

   $$P(X > 2) = 1 - e^{-1.08}\left(\frac{1.08^0}{0!} + \frac{1.08^1}{1!} + \frac{1.08^2}{2!}\right) = 0.0956$$

2) A particular disease is known to affect 0.3% of the population. A random sample of 300 people is taken.

   a) Write down the distribution of the data.
      $X \sim B(300,0.003)$
   b) Explain why the Poisson distribution would be a suitable approximation.
      n is large as n>50 and p is small as np=0.9<5
   c) Use the approximation to find the probability that fewer than four people in the sample have the disease.
      $X \sim Po(0.9)$

      $$P(X < 4) = e^{-0.9}\left(\frac{0.9^0}{0!} + \frac{0.9^1}{1!} + \frac{0.9^2}{2!} + \frac{0.9^3}{3!}\right) = 0.987$$

3) A machine produces lightbulbs and it is know that 99% of the bulbs produced are fault-free. A random sample of 200 bulbs is tested for faults. X is the number of faulty bulbs in the sample.
   a) Write down the distribution of X and a suitable approximate distribution; give a justification for your approximation.
      $X \sim B(300,0.01)$, we can use a Poisson approximation as n is large as n>50 and p is small as np=3<5 $X \sim Po(3)$
   b) Use the approximation to find $P(X \geq 1)$

      $$(X \geq 1) = 1 - e^{-3}\left(\frac{3^0}{0!}\right) = 0.950$$

Extension question
The random variable X follows the distribution $X \sim B(250, p)$.

Given that a Poisson approximation is used and $P(X = 3) = 0.15169$ and $P(X = 5) = 0.17475$ find the value of p to 2sf.

$$\frac{e^{-\lambda}\left(\frac{\lambda^5}{5!}\right)}{e^{-\lambda}\left(\frac{\lambda^3}{3!}\right)} = \frac{0.17475}{0.15169}$$

$$\frac{\lambda^2}{20} = 1.152020..$$

$$\lambda = 4.80004 \ldots$$

$$p = \frac{4.80004}{250} = 0.019(2sf)$$

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